

RESEARCHES IN NON-ASSOCIATIVE ALGEBRA.

By I. M. H. Etherington,  
Mathematical Institute, University of Edinburgh.

---

Thesis submitted to the University of Edinburgh for  
the degree of D. Sc., September 1941.

---



## CONTENTS

Introduction and historical note.

### PART ONE: NON-ASSOCIATIVE COMBINATIONS

Referred  
to by  
initials  
as

- I. "Non-associate powers and a functional equation."  
-Math. Gaz., 21 (1937) 36-39.
- II. "On non-associative combinations."  
-Proc. Roy. Soc. Edin., 59 (1939) 153-162. — N.C.
- III. "Some problems of non-associative combinations (1)."  
-Edinburgh Math. Notes, 32 (1941) 1-6.
- IV. (In collaboration with A. ERDELYI)  
"Some problems of non-associative combinations (2)."  
-Edinburgh Math. Notes, 32 (1941) 7-12.

### PART TWO: THE SYMBOLISM OF GENETICS

- V. "Non-associative algebra and the symbolism of genetics."  
-Proc. Roy. Soc. Edin., (B) 61 (1941) 24-42.
- VI. "Genetic algebras." - Ibid., 59 (1939) 242-258. — G.A.

### PART THREE: NON-ASSOCIATIVE LINEAR ALGEBRAS

- VII. "Duplication of linear algebras."  
-Proc. Edin. Math. Soc., (2) 6 (1941) 221-229.
- VIII. "Commutative train algebras of ranks 2 and 3."  
-Journ. Lond. Math. Soc., 15 (1940) 136-149. — C.T.A.
- IX. "Special train algebras."  
-Quart. Journ. Math., 12 (1941) 1-8. — S.T.A.
- X. "Some non-associative algebras in which the multiplication  
of indices is commutative."  
-Journ. Lond. Math. Soc., 16 (1941) 48-55.

Postscripts follow papers IV, IX and X.

---

Note:- From the point of view of the thesis as a whole, the  
fundamental sections are II. §§2,3 and VI. §§2-5.

---

## INTRODUCTION

I have frequently been asked by biologists for mathematical help in connection with their problems. I was working on one such problem when an algebraist, observing my work without knowing what it was about, remarked that I was apparently using hypercomplex numbers. I was considering a certain type of inheritance specified by formulae which could be regarded as forming the multiplication table of a non-associative linear algebra; and my calculations could be regarded as manipulations of hypercomplex numbers in this algebra, or in another algebra derived from it by a process which I later called "duplication!"

I then realised that there are many such "genetic algebras", representing different types of inheritance. They are in all cases non-associative as regards multiplication, though they can always be taken to be commutative. I found that a large class of genetic algebras (viz. those for "symmetrical inheritance" as defined in Paper VI, p. 2) possessed certain distinctive properties which seemed worthy of investigation for their own sake, and also for the sake of possible exploitation in genetics.

Part Three, the main part of this thesis, consists of four papers in which this investigation is given - or rather is begun, for there are a good many problems left untackled.

Part One consists of four papers (one written in collaboration with Dr A. Erdélyi) on some purely combinatory problems of non-associative algebra, suggested by the notations which I employed for products and powers in the genetic algebras. The combinatory theory is continued in the <sup>postscript to Part One, and in the</sup> concluding postscript which follows Paper X.

Part Two shows how genetic algebras arise and are manipulated. The multiplication table of a genetic algebra, the multiplication of hypercomplex numbers, and the above mentioned process of duplication, are simply a translation into symbols of the relevant essentials in the processes of inheritance; and the symbolism as a whole is a convenient shorthand for reckoning with combinations and statistical distributions of genetic types, enabling one to dispense with some of the verbal arguments and the chessboard diagrams commonly used for the same purpose. In paper VI the

treatment is made as general as possible with the object of showing the relationship between different genetic algebras and something of their structure; and the concepts to be discussed in Part Three are here defined. In Paper V, which was published later but mostly written earlier than VI, the explanation is given in very much simpler mathematical language (for it was intended to be read by geneticists), and with more attention to practical applications.

It can be explained very simply why multiplication in the genetic algebras is non-associative, that is to say

$$(AB)C \neq A(BC).$$

This statement is interpreted:- "If the offspring of A and B mates with C, the probability distribution of genetic types in the progeny will not be the same as if A mates with the offspring of B and C."

My symbolism was not essentially new: the novelty lay in its interpretation in terms of hypercomplex numbers. In fact it could be said that genetic algebras had been used by geneticists in a primitive way for quite a long time without having been recognised explicitly. Their explicit recognition is I believe more than a mere change of notation. Apart from greater brevity achieved in some applications, general theorems on linear algebras can be applied; transformations can be used which are quite meaningless genetically but which lead to genetically significant conclusions; and the use of an index notation and summation convention reduces the symbolism to manageable proportions when, with inheritance involving many genes, it threatens to become too heavy to handle.

Biological considerations were thus the root of these researches, and I intend to return to the genetical applications later; for I believe that genetic algebras may throw light on some deeper problems of genetics. I cannot at present give solid justification for this belief in the sense of having successfully tackled problems otherwise unsolved, and I therefore wish that this thesis may not be judged as a finished achievement in biological investigation; but may be judged primarily as a contribution to algebra, suggested by biological problems, and perhaps having possibilities of application beyond the simple ones so far demonstrated.



## HISTORICAL NOTE

I have not studied the history of non-associative algebra systematically; and this note is therefore no more than a rough sketch, emphasising work which has attracted my attention as being in some way related to my own.

While most of the classical work on linear algebras has been confined to associative algebras, non-associative ones are by no means unknown. Early in the history of the subject De Morgan<sup>1</sup> and Cayley<sup>2</sup> studied simple types. Special kinds of non-associative algebras (i.e. algebras in which some special property takes the place of the associative law of multiplication) have been studied extensively; particularly Lie algebras<sup>3</sup>, important for the representation of continuous groups; also alternative rings<sup>4</sup>; Jordan's "r-number" algebras, designed to suggest a possible generalisation of quantum mechanics<sup>5</sup>; "quasi-associative" and "Jacobian" algebras<sup>6</sup>. Wedderburn<sup>7</sup> and Dickson<sup>8</sup> discussed non-associative algebras on more general lines, dealing with their ideal structure and invariantive classification. This investigation has been continued by numerous algebraists, mainly in America, leading to a more thorough axiomatic investigation of various kinds of abstract non-associative algebraical systems, notably by Ore and his collaborators<sup>9</sup>.

The occurrence of a non-associative linear algebra in the simplest case of mendelian inheritance was pointed out by Glivenko<sup>10</sup>.

For a bibliography of non-associative combinatory theory, see Papers II and III.

1. A. De Morgan, Camb. Phil. Trans., 8 (1844) 241-254.
2. A. Cayley, Amer. Journ. Math., 4 (1881) 293-296; Proc. Lond. Math. Soc., (1) 15 (1884) 185-197; Quart. Journ. Math., 22 (1887) 270-308.
3. For references, see Jacobson, Ann. Math. 38 (1937) 508-517. The property expressed by my formula (7.5) in Paper VIII is characteristic of a Lie algebra.
4. M. Zorn, Abh. Math. Sem. Hamburg, 8 (1931) 123-147; 9 (1933) 395-402, etc.
5. P. Jordan, Gottingen Nachrichten, 1932, 569; 1933, 209; P. Jordan, J. v. Neumann, E. Wigner, Ann. Math., 35 (1934) 29-64. See footnote to Paper VIII, p. 138.
6. G. C. Moisil, Ann. Sci. Jassy, 20 (1935) 10-32. Jacobian algebras are those having the property mentioned in reference 3.
7. J. H. M. Wedderburn, Proc. Lond. Math. Soc., (2) 6 (1908) 77-118.
8. L. E. Dickson. See references given in Papers VI, p. 258, and VIII, p. 13; also Duke Math. Journ., 1 (1935) 113-125.
9. B. A. Hausmann and O. Ore, Amer. Journ. Math., 59 (1937) 983-1003. D. C. Murdoch, ibid., 61 (1939) 509-  
D. C. Murdoch and O. Ore, ~~Amer~~ ibid., 63 (1941) 73-86.
10. V. Glivenko, Moscow Acad. Sci. C.R., 4 (1936) 385-386.

PART ONE

NON-ASSOCIATIVE COMBINATIONS

NON-ASSOCIATE POWERS AND A FUNCTIONAL  
EQUATION.

BY I. M. H. ETHERINGTON.

(Most of this note was anticipated by other writers, but the graphical treatment of the functional equation is new. This equation was discussed analytically by Wedderburn in terms of functions of a complex variable. For references, see *Paper II* pp. 159-160.)

It is necessary in a non-associative algebra to distinguish the possible interpretations of a power  $x^n$ . In a non-commutative non-associative algebra  $x^2$  is unique;  $x^3$  can mean  $xx^2$  or  $x^2x$ ;  $x^4$  can mean  $x \cdot xx^2$ ,  $x \cdot x^2x$ ,  $x^2x^2$ ,  $xx^2 \cdot x$  or  $x^2x \cdot x$ ;  $x^5$  has 14 interpretations;  $x^6$  has 42; and so on. In a commutative non-associative algebra, the possible interpretations are fewer:  $x^3$  is unique;  $x^4$  can mean  $xx^3$  or  $x^2x^2$ ;  $x^5$  can mean  $x \cdot xx^3$ ,  $x \cdot x^2x^2$  or  $x^2x^3$ ;  $x^6$  has 6 interpretations; and so on. The problem considered here is: how many meanings are there for  $x^n$  (A) in a general non-commutative non-associative algebra? (B) in a general commutative non-associative algebra? The answer to (A) is  $\frac{2(2n-3)!}{n!(n-2)!}$ . I am not able to find any such simple formula for (B).

It may be observed that (A) is equivalent to asking how many meanings there are in general in ordinary algebra for  $x_1 : x_2 : \dots : x_n$ , where each colon stands for any non-commutative non-associative process, such as subtraction, division or exponentiation.

(A) *The number of interpretations of  $x^n$  in a general non-commutative non-associative algebra.*

Let  $a_n$  denote the required number of interpretations. Since  $x^n$  can arise in any of the ways  $xx^{n-1}$ ,  $x^2x^{n-2}$ ,  $x^3x^{n-3}$ , ...,  $x^{n-1}x$ , we have at once for  $n > 1$

$$a_n = a_1 a_{n-1} + a_2 a_{n-2} + a_3 a_{n-3} + \dots + a_{n-1} a_1.$$

The left-hand side is the coefficient of  $x^n$  in

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots \text{ ad inf. ;}$$

while the right-hand side, for  $n > 1$ , is the coefficient of  $x^n$  in  $\{F(x)\}^2$ .

\* Therefore for sufficiently small values of  $x$  (assuming the absolute convergence of the series for  $F(x)$ )

$$F(x) = x + \{F(x)\}^2, \dots \dots \dots (1)$$

whence

$$F(x) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4x}.$$

The minus sign must be taken because  $F(0) = 0$ . So

$$F(x) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4x} = \frac{1}{2} - \frac{1}{2} (1 - 2x + \frac{1}{2} (-\frac{1}{2}) \cdot 16x^2 - \dots), \dots \dots (2)$$

\* the assumption of absolute convergence being justified if  $|x| < \frac{1}{4}$ .

\* The consideration of convergence is not essential.

Hence  $a_n$  = coefficient of  $x^n$

$$= -\frac{1}{2 \cdot n!} \cdot \frac{1}{2} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \dots \left(-\frac{2n-3}{2}\right) \cdot (-4)^n.$$

$$\text{If } n > 1, \text{ this} = [1 \cdot 3 \cdot 5 \dots (2n-3)] 2^{n-1} \div n! = \frac{2(2n-3)!}{n!(n-2)!} = \frac{(2n-2)!}{(n-1)! n!},$$

a formula which is easily checked for small values of  $n$ .

(B) *An attempt to find the number of interpretations of  $x^n$  in a general commutative non-associative algebra.*

The attempt fails because the general term of the expansion corresponding to (2) cannot be written down. However, as the functional equation (4) which replaces (1) is interesting on its own account, it is perhaps worth while giving the details.

Let  $b_n$  denote the required number of interpretations, and let

$$f(x) = -1 + b_1x + b_2x^2 + b_3x^3 + \dots \text{ ad inf.} \dots\dots\dots(3)$$

The insertion of  $-1$  simplifies the functional equation which arises.

Since  $b_n < a_n$ , the series is absolutely convergent when  $|x| < \frac{1}{4}$ ; so  $f(x)$  may be formally squared.

Now  $x^n$  can be formed in any of the ways  $xx^{n-1}, x^2x^{n-2}, x^3x^{n-3}, \dots$

This series terminates with  $x^{\frac{n-1}{2}}x^{\frac{n+1}{2}}$  if  $n$  is odd and  $>1$ , with  $x^{\frac{n}{2}}x^{\frac{n}{2}}$  if  $n$  is even. Hence, for  $n > 1$ ,

$$\begin{aligned} b_{2n-1} &= b_1b_{2n-2} + b_2b_{2n-3} + \dots + b_{n-1}b_n \\ &= \text{half the coefficient of } x^{2n-1} \text{ in } \{f(x)+1\}^2; \\ b_{2n} &= b_1b_{2n-1} + b_2b_{2n-2} + \dots + b_{n-1}b_{n+1} + \frac{1}{2}b_n(b_n+1) \\ &= \text{half the coefficient of } x^{2n} \text{ in } \{f(x)+1\}^2 + f(x^2). \end{aligned}$$

So the expansions of the functions  $f(x)$  and  $\frac{1}{2}[\{f(x)+1\}^2 + f(x^2)]$  agree from the terms in  $x^3$  onwards. These expansions begin respectively  $-1+x+x^2+x^3+\dots$  and  $-\frac{1}{2}+x^2+x^3+\dots$ . It follows that

$$f(x)+1-x = \frac{1}{2}[\{f(x)+1\}^2 + f(x^2)] + \frac{1}{2},$$

that is,

$$\{f(x)\}^2 + f(x^2) + 2x = 0, \dots\dots\dots(4)$$

or

$$f(x) = -\sqrt{-2x - f(x^2)}. \dots\dots\dots(5)$$

When  $x$  is small, we know from (3) that, to a first approximation,  $f(x) = -1 = f(x^2)$ ; substituting this in the right-hand side of (5), we get the second approximation  $f(x) = -\sqrt{-2x+1}$ ,  $f(x^2) = -\sqrt{-2x^2+1}$ ; similarly the third approximation is  $f(x) = -\sqrt{-2x+\sqrt{-2x^2+1}}$ , and so on. Hence

$$f(x) = \lim_{N \rightarrow \infty} -\sqrt{-2x + \sqrt{-2x^2 + \sqrt{-2x^4 + \dots + \sqrt{-2x^{2^N} + 1}}}}, \dots\dots(6)$$

where each  $\sqrt{\phantom{x}}$  sign covers all that follows it.

Abandoning the attempt to find  $b_n$  explicitly by expanding this generating function, let us consider the function itself and the various other functions which satisfy equation (4) but not (3).

\*

Suppose we assign an arbitrary value to  $f(x_0)$ . Then from (4) we can calculate in succession  $f(x_0^2), f(x_0^4), f(x_0^8), \dots$ ; and the (possibly infinite) limit of this sequence gives either  $f(0)$  or  $f(\infty)$  according as  $|x_0| < \text{or} > 1$ . This shows that the functions which satisfy (4) are of two kinds, those for which  $f(0)$  exists and those for which  $f(\infty)$  exists. In fact it will be found that no curve of the system  $y=f(x)$  crosses the boundaries  $x=\pm 1$ .

Consider first the functions for which  $f(0)$  exists. These are of three types: for (4) gives  $\{f(0)\}^2 + f(0) = 0$ , so that

$$f(0) = 0, -1 \text{ or } -\infty. \dots\dots\dots (7)$$

When  $x$  is small, let the first approximation be  $f(x) = kx^\alpha$ .

Substituting in (5), we get as the second approximation

$$f(x) = -\sqrt{-2x - kx^{2\alpha}}, \quad f(x^2) = -\sqrt{-2x^2 - kx^{4\alpha}}.$$

For  $f(x^2)$  to be real the term in  $x^{4\alpha}$  must predominate, so  $\alpha < \frac{1}{2}$ ; and in order that the two approximations for  $f(x)$  should agree,  $k = -1$ . Iterating as before,

$$f(x) = \lim_{N \rightarrow \infty} -\sqrt{-2x + \sqrt{-2x^2 + \sqrt{-2x^4 + \dots + \sqrt{-2x^{2^N} + x^{2^{N+1}\alpha}}}}}, \dots (8)$$

where each  $\sqrt{\phantom{x}}$  sign covers all that follows it. The three types (7) correspond to  $\frac{1}{2} > \alpha > 0$ ,  $\alpha = 0$ ,  $\alpha < 0$ ; and  $\alpha = 0$  gives (6).

The functions for which  $f(\infty)$  exists are found in precisely the same way. Assuming  $f(x) \sim kx^\alpha$  for large  $x$ , we reach by iteration the same formula (8), but now  $\alpha > \frac{1}{2}$ .

If  $x > 0$ , the right-hand side of (8) has a factor  $\sqrt{x}$ . Writing  $f(x) = Y\sqrt{x}$ , the formula becomes

$$Y = -\lim_{N \rightarrow \infty} \theta^{N+1} (x^{2^{N+1}\alpha - 2^N}), \dots\dots\dots (9a)$$

where  $\theta$  is the operation defined by  $\theta t = \sqrt{-2+t}$ . Inverting (9a),

$$x = \lim_{N \rightarrow \infty} (\phi^{N+1} Y)^{\frac{1}{2\alpha-1} \cdot \frac{1}{2^N}}, \dots\dots\dots (10a)$$

where  $\phi t = t^2 + 2$ . Similarly for  $x < 0$ , writing  $f(x) = Z\sqrt{-x}$ ,

$$Z = -\lim_{N \rightarrow \infty} \sqrt{2 + \theta^N x^{2^{N+1}\alpha - 2^N}}, \dots\dots\dots (9b)$$

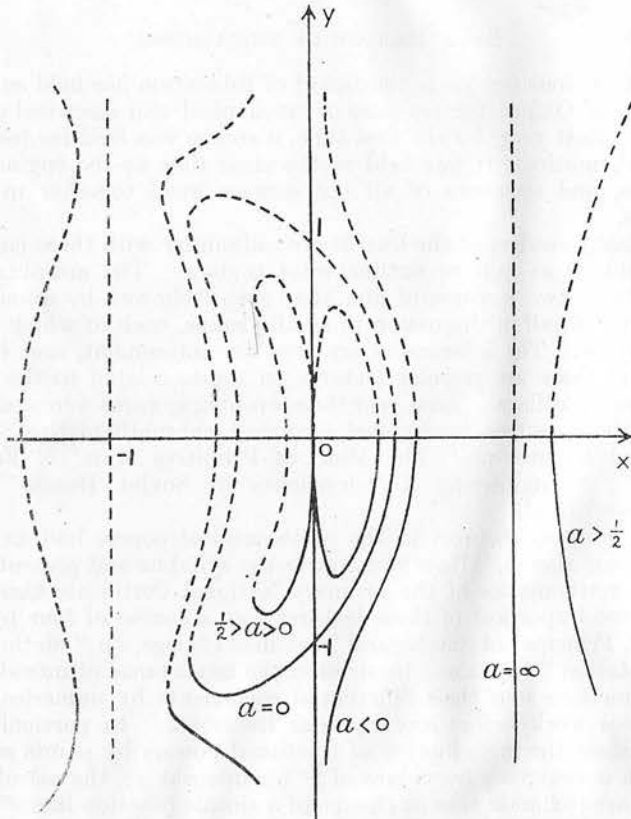
$$x = \lim_{N \rightarrow \infty} \{\phi^N (Z^2 - 2)\}^{\frac{1}{2\alpha-1} \cdot \frac{1}{2^N}}, \dots\dots\dots (10b)$$

The curves  $y=f(x)$  can be traced from these formulae (8)-(10) combined with (4). The various types are shown by the continuous lines in the figure. When  $\alpha < 0$  or  $> \frac{1}{2}$ , the curve has two branches. If  $f(x^2)$  in (4) is taken to mean "one value of  $f(x^2)$ ", then all the radicals in (8) and (9) can be taken as "plus or minus". (Most of the radicals have to be taken positively to yield a real value of  $f(x)$ .)

\* The somewhat loose argument which follows here can be rigorised without difficulty.



With this generalisation the curves continue as shown by the dotted lines in the figure. The zeros of  $f(x)$  can be calculated from equations (10) by putting  $Y=0, Z=0$ . These equations also show that



for  $\alpha=\infty, x=\pm 1$ ; and that for  $\alpha=\frac{1}{2}$  the curve becomes the line at infinity plus the origin. Actually, when  $\alpha=\frac{1}{2}$ , the process of iteration terminates and gives the solution  $f(x)=kx^{\frac{1}{2}}$ , where  $k$  is one of the (imaginary) roots of  $k^2+k+2=0$ .

REPRINT FROM THE  
PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH

SESSION 1938-1939

VOL. LIX—PART II—(No. 15)

XV.—On Non-Associative Combinations. By I. M. H. Etherington, B.A.(Oxon.), Ph.D.(Edin.), Mathematical Institute, University of Edinburgh.

(MS. received October 18, 1938. Revised MS received March 24, 1939.  
Read May 1, 1939.)

CONTENTS.

|                           | PAGE |                               | PAGE |
|---------------------------|------|-------------------------------|------|
| § 1. INTRODUCTION         | 153  | § 3. CLASSIFICATION OF SHAPES | 155  |
| § 2. ARITHMETIC OF SHAPES | 154  | § 4. ENUMERATION OF SHAPES    | 159  |

§ 1. INTRODUCTION.

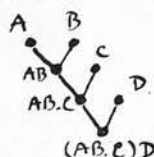
NUMEROUS combinatory problems arise in connection with a set of elements subject to a non-associative process of composition—let us say of multiplication—commutative or non-commutative.

Non-associative products may be classified according to their *shape*. By the shape of a product I mean the manner of association of its factors without regard to their identity. Shapes will be called *commutative* or *non-commutative* according to the type of multiplication under consideration. Thus if multiplication is non-commutative, the products  $(AB.C)D$  and  $(BA.C)D$  are distinct but have the same shape, while  $D(AB.C)$  has a different shape. The three expressions, however, have the same commutative shape. I confine attention to products (like these) in which the factors are combined only two at a time.

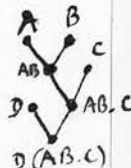
In § 2 I define addition and multiplication of shapes, and show that they may be regarded as the “positive integers” of a kind of non-associative arithmetic. With commutative multiplication this provides a convenient numerical notation by which shapes of great complexity can be easily specified.

A non-associative product or shape may be visualised as a *pedigree*<sup>\*</sup>, by which I mean a *tree* (Cayley, 1857) which (going from the root upwards, i.e. from the product to its factor elements) bifurcates at every knot (Cayley, 1859). Trees in general may quantifurcate arbitrarily at the knots, representing a more general kind of non-associative assemblage,<sup>†</sup> which was also considered abstractly by Schröder (1870). A four-fold classification of shapes, arising partly out of this representation, is discussed in § 3.

\* E.g., the pedigree of  $(AB.C)D$  is



, that of  $D(AB.C)$  is



† See, e.g., Paper III, Figure 2.

Enumerative problems connected with non-associative combinations have been considered from various points of view by Catalan (1838, p. 515; etc.), Rodrigues (1838), Binet (1839), Schröder (1870): see Netto (1901, §§ 122-128) for a summary of their work; also by Cayley (1857, etc.), Wedderburn (1922). Some further enumerations are discussed here (§ 4); in particular, with the aid of the concept of *mutability*, defined in § 3, it is shown that the commutative and non-commutative cases can be treated simultaneously. Thus equation (33) below, with  $y$  put equal to 1, yields the known formulæ (25), (27) for the commutative case; putting  $y=2$ , the known results (24), (26), (28) follow for the non-commutative case.

## § 2. ARITHMETIC OF SHAPES.

To eliminate brackets in writing non-associative products, it is convenient to use groups of dots to separate the factors when necessary, fewness of dots implying precedence in multiplication. Thus  $A : BC \cdot AD^2 : E$  means  $A\{(BC)(ADD)\}E$ . (The notation is due to Peano.)

\* Products and shapes in which the factors are absorbed one at a time (e.g.  $A : BC \cdot D : E$ ) will be called *primary*. The shapes generated by repeated squaring of an element, and products having such a shape (e.g.  $AB \cdot CD : EF \cdot GH$ ), will be called *plenary*. It will be seen in § 3 that all other shapes are in a sense intermediate between these two extremes.

For the moment, confine attention to the case of commutative multiplication, where a primary shape is unique when the number of factors  $\delta$  is given. A power having this shape will be denoted  $X^\delta$ : e.g.  $X^4$  means  $XX \cdot X : X$ . All other powers can be represented by suitably partitioning the index, using brackets when necessary, with the following conventions: the product of two powers of the same element is indicated as a sum in the index, a power of a power as a product in the index, and an iterated power as a power in the index. Thus:

$$\begin{aligned} X^{2+3} &= X^2 X^3, \\ X^{2 \cdot 3} &= (X^2)^3, & X^{3 \cdot 2} &= (X^3)^2, \\ X^{2^2} &= ((X^2)^2)^2, & X^{3^2} &= (X^3)^3, \\ X^{(2 \cdot 2 + 3) + (2 + 3)^2} &= X^2 X^2 \cdot X^3 : X^2 X^3 \cdot X^2 X^3 : X^2 X^3 \cdot X^2 X^3. \end{aligned}$$

Addition of indices, since it reflects non-associative multiplication of powers, is commutative but non-associative. On the other hand, multiplication of indices is non-commutative (as seen above), but associative, since  $X^{ab \cdot c}$  and  $X^{a \cdot bc}$  both mean  $((X^a)^b)^c$ , which can therefore be written  $X^{abc}$  unambiguously. This becomes  $X^{a^3}$  when  $a=b=c$ ; and similarly with any number of factors in the index.

\* Examples:-

Primary shapes:



Plenary shapes:



Those primary shapes in which the successive factors are added always on the right [or always on the left] are called later principal [Cf. Papers V, p. 28; VI, p. 143]; for example: When multiplication is commutative, principal and primary mean the same.

Further,  $X^{a(b+c)}$  means  $(X^a)^{b+c}$ , i.e.  $(X^a)^b(X^a)^c$ , which is the same as  $X^{ab+ac}$ . Hence in the arithmetic of the indices

$$a(b+c) = ab + ac.$$

But in general

$$(b+c)a \neq ba + ca,$$

since  $(X^b X^c)^a$  is not the same as  $(X^b)^a (X^c)^a$ . We may say therefore that in the arithmetic of the indices <sup>Pre-</sup>multiplication is ~~pre~~distributive ~~with~~ addition, but not in general postdistributive multiplication.

In these arguments  $a, b, c$  can be any expressions standing for complicated powers: they are not restricted to being simple integers indicating primary powers.

The notation provides an arithmetical method of specifying commutative shapes; for now the *shape*  $s$  of any commutative non-associative product can be redefined as the index of the corresponding power obtained by equating all the factors. The product  $AB.C^2:D$ , for instance, has the same shape as the power  $(X^2)^2 X = X^{2 \cdot 2 + 1}$ , namely  $s = 2 \cdot 2 + 1$ .

Consider what addition and multiplication of shapes mean when we are dealing with products in general instead of powers. Let  $\Pi_1, \Pi_2$  be any two products with shapes  $s_1, s_2$ . Then  $s_1 + s_2$  is the shape of the product  $\Pi_1 \Pi_2$ , while  $s_1 s_2$  is the shape of the product formed by substituting  $\Pi_1$  for each of the factor elements of  $\Pi_2$ .

The procedure of this § may be described as a representation of the set of all commutative non-associative continued products formed from given elements on a non-associative arithmetic, whose integers are commutative shapes  $a, b, c, \dots$  with the rules of combination

$$\left. \begin{array}{l} a+b=b+a, \quad ab \cdot c = a \cdot bc, \quad a(b+c) = ab + ac, \\ ab \neq ba, \quad (a+b)+c \neq a+(b+c), \quad (b+c)a \neq ba + ca. \end{array} \right\} \quad (I)$$

A similar representation is possible when multiplication of the original elements is non-commutative as well as non-associative. It is reflected as non-commutative addition of shapes, the other rules of combination (I) being unchanged. But the numerical specification of non-commutative shapes of increasing complexity rapidly becomes very complicated; to simplify it, some convention is required for distinguishing the  $2^{\delta-2}$  distinct primary shapes of any given degree  $\delta (> 1)$ .

### § 3. CLASSIFICATION OF SHAPES.

Shapes  $s$  will be classified by their *degree*  $\delta(s)$ , *altitude*  $\alpha(s)$ , and *mutability*  $\mu(s)$ . Non-commutative shapes will be further classified by

\* Thus for example the podigrees of the commutative shapes  
 $2, \quad 3, \quad 2 \cdot 3, \quad 3 \cdot 2, \quad 2+3$   
 are

the commutative shapes with which they are *conformal*. These terms will now be defined.

The *degree*  $\delta$  of a shape  $s$  means the number of factor elements in a product having this shape. It may be reckoned by evaluating  $s$  as if it were an integer in ordinary arithmetic.

Two non-commutative shapes  $s_1, s_2$ , which become the same shape  $s$  when multiplication is regarded as commutative, will be called *conformal* with each other and with  $s$ . Write  $s_1 \sim s_2$  to indicate this. With commutative shapes,  $s_1 \sim s_2$  means the same as  $s_1 = s_2$ , a commutative shape being conformal only with itself. The word is also applicable to products whose shapes are conformal. Thus

$$AB.C : D, \quad A.BC : D, \quad A : BC.D, \quad A : B.CD$$

and their shapes

$$(2+1)+1, \quad (1+2)+1, \quad 1+(2+1), \quad 1+(1+2),$$

are all conformal with the commutative power  $A^4$  and its shape 4.

Let shapes be depicted as pedigrees (§ 1). Any non-associative product is then, so to speak, "descended from" its factors. The number of "generations" preceding the product itself is its *altitude*  $a$  (Cayley, 1875). At each knot in the pedigree two factors are united; the total number of knots is thus  $\delta - 1$ . Let a knot be called *balanced* if its two factors are conformal: then the number of unbalanced knots in a pedigree will be called its *mutability*  $\mu$ .\* The various terms defined may be applied indiscriminately to the product, shape or pedigree.

If the mutability of any shape  $s$  (commutative or not) is  $\mu$ , then there are evidently just  $2^\mu$  distinct non-commutative shapes which will become the same as  $s$  when multiplication is commutative. So  $\mu$  could be defined alternatively as the logarithm to base 2 of the number of conformal non-commutative shapes.

If  $s_1 \sim s_2$ , then evidently

$$\delta(s_1) = \delta(s_2), \quad a(s_1) = a(s_2), \quad \mu(s_1) = \mu(s_2). \quad (2)$$

The following formulæ are easily proved,  $r$  and  $s$  being any shapes, commutative or non-commutative, and  $v$  an ordinary positive integer:—

$$\delta(r+s) = \delta(r) + \delta(s), \quad (3)$$


$$\delta(rs) = \delta(r)\delta(s), \quad (4)$$

$$\delta(s^v) = \delta(s)^v; \quad (5)$$

$$a(r+s) = 1 + a(r) \quad \text{or} \quad 1 + a(s) \quad \text{according as} \quad a(r) \geq \quad \text{or} \quad \leq a(s), \quad (6)$$

$$a(rs) = a(r) + a(s), \quad (7)$$

$$a(s^v) = va(s); \quad (8)$$

\* Consider for example the commutative shape  $4+2$ . Its pedigree is . The three knots marked in red are unbalanced. Hence  $\mu(4+2) = 3$ .



$$\mu(r+s) = 2\mu(s) \quad \text{if } r \sim s, \quad (9)$$

$$= 1 + \mu(r) + \mu(s) \quad \text{if not,} \quad (10)$$

$$\mu(rs) = \delta(s)\mu(r) + \mu(s), \quad (11)$$

$$\mu(s^n) = (1 + \delta + \delta^2 + \dots + \delta^{n-1})\mu(s), \quad (12)$$

where

$$\delta = \delta(s).$$

The last result is proved by induction from the preceding one. It may also be written

$$\frac{\mu(s^n)}{\mu(s)} = \frac{\tau(s^n)}{\tau(s)}, \quad (13)$$

where

$$\tau = \delta - 1.$$

The degree, altitude and mutability can now be readily calculated for any given shape specified numerically. The table below gives all commutative shapes for which  $a \leq 4$ ,  $\delta \leq 6$ , and their pedigrees.

TABLE OF COMMUTATIVE SHAPES.

| $a$ . | $\delta$ . | $\mu$ . | $s$ .          |
|-------|------------|---------|----------------|
| 0     | 1          | 0       | 1              |
| 1     | 2          | 0       | 2              |
| 2     | 3          | 1       | 3              |
| 2     | 4          | 0       | 2.2            |
| 3     | 4          | 2       | 4              |
| 3     | 5          | 1       | 2.2+1          |
| 3     | 5          | 2       | 3+2            |
| 3     | 6          | 1       | 2.3            |
| 3     | 6          | 2       | 3.2            |
| 3     | 7          | 2       | 2.2+3          |
| 3     | 8          | 0       | 2 <sup>3</sup> |
| 4     | 5          | 3       | 5              |
| 4     | 6          | 2       | (2.2+1)+1      |
| 4     | 6          | 3       | (3+2)+1, 4+2   |
| Etc.  |            |         |                |

As the table suggests, we cannot construct a shape with  $a$ ,  $\delta$ ,  $\mu$  assigned arbitrarily. Certain relations must be satisfied, namely:

$$2^a \geq \delta \geq a+1; \text{ i.e. } \delta-1 \geq a \geq \log_2 \delta. \quad (14)$$

$$\delta \geq \mu+2, \text{ except when } \delta=1. \quad (15)$$

$$\mu \leq 3 \cdot 2^{a-3} - 1; \text{ i.e. } a \geq 3 + \log_2 \frac{\mu+1}{3}, \text{ except when } a < 3. \quad (16)$$

$$\delta \text{ is expressible as the sum of } \mu+1 \text{ powers of } 2, \text{ not all alike if } \mu > 0. \quad (17)$$

(14) and (15) are easily proved by consideration of pedigrees. At one extreme, the equality  $\delta = a+1$  holds only when  $s$  is primary; and the

same is true of  $\delta = \mu + 2$ . Similarly at the other extreme,  $\delta = 2^a$ ,  $\mu = 0$  occur when and only when  $s$  is plenary.

(17) is proved by induction from (3), (5), (9), (10); it being noted that when  $\mu = 0$ ,  $s$  is of the form  $2^a$  (plenary); when  $\mu = 1$ ,  $s = 2^a + 2^b$  ( $a \neq b$ ); and that two like powers of 2 can be combined if desired into a single power of 2.

To prove (16), let  $\mu_a$  be the greatest possible mutability for a shape whose altitude  $a$  is given; it will be shown that for  $a \geq 3$

$$\mu_a = 3 \cdot 2^{a-3} - 1.$$

In view of (2) it will be sufficient to consider only commutative shapes. By inspection of the table of commutative shapes,

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = 2.$$

Now (see (6)) any shape of altitude  $a + 1$  is necessarily the sum of two shapes, one of altitude  $a$  and one of altitude  $\beta \leq a$ . By (9), (10),  $\mu_{a+1}$  must be expressible either as  $2\mu_a$  or as  $1 + \mu_a + \mu_\beta$ . Since  $\mu_1 = 0$ ,  $\mu_2 = 1$ , it follows that

$$\mu_{a+1} > \mu_a \quad \text{for } a > 0,$$

so that  $\mu_a$  increases monotonically with  $a$ .

Now let  $a$  be any altitude (e.g.  $a = 3$ ) for which there exist at least three distinct shapes  $s_1, s_2, s_3$  with the maximum mutability  $\mu_a$ . Then

$$\mu(s_1) = \mu(s_2) = \mu(s_3) = \mu_a > \mu(s),$$

where  $s$  is any shape of lower altitude. Hence for the altitude  $a + 1$  also there will exist at least three distinct shapes of maximum mutability; namely,

$$s_1 + s_2, \quad s_2 + s_3, \quad s_3 + s_1,$$

with the mutability given by (10)

$$\mu_{a+1} = 1 + 2\mu_a. \quad (a \geq 3.)$$

It follows that

$$1 + \mu_{a+1} = 2(1 + \mu_a).$$

But

$$1 + \mu_3 = 3,$$

whence

$$1 + \mu_a = 3 \cdot 2^{a-3},$$

or

$$\mu_a = 3 \cdot 2^{a-3} - 1 \quad \text{if } a \geq 3.$$

This proves (16).

It will be seen that the equality in (16) is attained by  $N_a$  commutative shapes of altitude  $a$ , where

$$N_{a+1} = \frac{1}{2}N_a(N_a - 1), \quad N_3 = 4. \quad (16a)$$

§ 4. ENUMERATION OF SHAPES.

Let  $a_\delta, p_a$  be the numbers of possible shapes of given degree  $\delta$  and of given altitude  $a$  respectively, when multiplication is non-commutative and non-associative; and let  $b_\delta, q_a$  be the corresponding numbers when multiplication is commutative and non-associative. Evidently

$$a_1 = b_1 = p_0 = q_0 = 1.$$

Remembering (3) and (6), and considering the different ways in which shapes of given degree or altitude can be formed from those of lower degree or altitude, we obtain the formulæ:

$$a_\delta = a_1 a_{\delta-1} + a_2 a_{\delta-2} + a_3 a_{\delta-3} + \dots + a_{\delta-1} a_1, \quad (18)$$

$$\left. \begin{aligned} b_{2\delta-1} &= b_1 b_{2\delta-2} + b_2 b_{2\delta-3} + \dots + b_{\delta-1} b_\delta, \\ b_{2\delta} &= b_1 b_{2\delta-1} + b_2 b_{2\delta-2} + \dots + b_{\delta-1} b_{\delta+1} + \frac{1}{2} b_\delta (b_\delta + 1), \end{aligned} \right\} \quad (19)$$

$$p_{a+1} = 2p_a(p_0 + p_1 + p_2 + \dots + p_{a-1}) + p_a^2, \quad (20)$$

$$q_{a+1} = q_a(q_0 + q_1 + q_2 + \dots + q_{a-1}) + \frac{1}{2} q_a(q_a + 1). \quad (21)$$

For  $\delta = 1, 2, 3, \dots$  and  $a = 0, 1, 2, \dots$  the sequences start:

$$\begin{aligned} a_\delta &= 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \dots \\ b_\delta &= 1, 1, 1, 2, 3, 6, 11, 23, 46, 98, \dots \\ p_a &= 1, 1, 3, 21, 651, 457653, 210065930571, \dots \\ q_a &= 1, 1, 2, 7, 56, 2212, 2595782, \dots \end{aligned}$$

Let

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_\delta x^\delta + \dots \quad (22)$$

and

$$f(x) = -1 + b_1 x + b_2 x^2 + \dots + b_\delta x^\delta + \dots \quad (23)$$

The following results are known:—

$$F(x)^2 - F(x) + x = 0, \quad (24)$$

$$f(x)^2 + f(x^2) + 2x = 0; \quad (25)$$

$$F(x) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4x}, \quad (26)$$

$$f(x) = \lim_{n \rightarrow \infty} -\sqrt{-2x} + \sqrt{-2x^2} + \sqrt{-2x^4} + \dots + \sqrt{-2x^{2^n}} + 1, \quad (27)$$

where in (27) each  $\sqrt{\quad}$  covers all that follows it;

$$a_\delta = \frac{(2\delta - 2)!}{(\delta - 1)! \delta!} = \frac{1}{\delta} \cdot {}^{2\delta-2}C_{\delta-1}. \quad (28)$$

Of those formulæ, (18), (28) were given by Catalan (1838). (Catalan pointed out that  $a_\delta$  is the number of ways in which a convex polygon of  $\delta + 1$  sides can be divided up into triangles by diagonals. (28), as a consequence of (18) with  $a_1 = 1$ , was first established from this point of view, and was known to other writers, apparently first to Euler. Several papers on this topic appear in the *Journ. de Math.*, 1838-39.) Binet (1839)

introduced the generating function (22), and deduced (24), (26), (28) from (18). The calculations were repeated by Cayley (1859) from the pedigree point of view; by Schröder (1870); also by Wedderburn (1922), who discussed as well the commutative case, obtaining (19), (25), (27), and made a special study of the functional equation (25) and its more general solutions. (Cf. Etherington, 1937.)

It will now be shown that by introducing mutability we can discuss the commutative and non-commutative cases simultaneously and obtain a more general functional equation (33) which includes the two equations (24), (25) as special cases.

Let  $c_{\delta\mu}$  be the number of possible commutative shapes of given degree  $\delta$  and mutability  $\mu$ , so that the corresponding number of non-commutative shapes will be

$$n_{\delta\mu} = 2^\mu c_{\delta\mu}. \quad (29)$$

Then  $n_{\delta\mu}$ ,  $c_{\delta\mu}$  are defined for all integer values of  $\delta$ ,  $\mu$  with  $\delta \geq 1$ ,  $\mu \geq 0$ . For all other values of  $\delta$  and  $\mu$ , let  $n_{\delta\mu}$ ,  $c_{\delta\mu}$  be defined as zero.

Consider with the aid of (3), (9), (10) the different ways in which a non-commutative shape  $s$  of degree  $\delta$  and mutability  $\mu$  can be formed. Excluding  $\delta = 1$ ,  $\mu = 0$ ,  $s = 1$ ,  $s$  must be of the form  $s_1 + s_2$  where, by (3),

$$\delta(s_1) + \delta(s_2) = \delta.$$

If (10) held in all cases, we should have

$$1 + \mu(s_1) + \mu(s_2) = \mu,$$

and consequently

$$n_{\delta\mu} = \sum_{i,j,l,m} n_{il} n_{jm} \quad (i+j=\delta, \quad 1+l+m=\mu).$$

Subtracting the cases to which (10) does not apply, and adding those to which (9) does, we get as the correct formula

$$\left. \begin{aligned} n_{\delta\mu} &= \sum_{i,j,l,m} n_{il} n_{jm} - 2^{\frac{1}{2}(\mu-1)} n_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)} + 2^{\frac{1}{2}\mu} n_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i+j=\delta, \quad l+m=\mu-1, \quad \delta \neq 1. \\ \text{Also} \quad & n_{10} = 1. \end{aligned} \right\} \quad (30)$$

Putting  $n_{\delta\mu} = 2^\mu c_{\delta\mu}$ , and removing the factor  $2^\mu$ ,

$$\left. \begin{aligned} c_{\delta\mu} &= \frac{1}{2} \left( \sum_{i,j,l,m} c_{il} c_{jm} - c_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)} \right) + c_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i+j=\delta, \quad l+m=\mu-1, \quad \delta \neq 1. \\ \text{Also} \quad & c_{10} = 1. \end{aligned} \right\} \quad (31)$$

Now let

$$f(x, y) = \sum_{\delta, \mu} c_{\delta\mu} x^\delta y^\mu. \quad (32)$$

Substituting (31) in (32), we obtain the functional equation

$$f(x, y) = x + \frac{1}{2}y\{f(x, y)\}^2 + (1 - \frac{1}{2}y)f(x^2, y^2). \quad (33)$$

Now, from the definitions of  $a_\delta$ ,  $b_\delta$ ,  $c_{\delta\mu}$ ,  $n_{\delta\mu}$ ,

$$a_\delta = \sum_{\mu} n_{\delta\mu} = \sum_{\mu} 2^\mu c_{\delta\mu}, \quad b_\delta = \sum_{\mu} c_{\delta\mu}.$$

Consequently, comparing (22), (23), (32),

$$F(x) = f(x, 2), \quad 1 + f(x) = f(x, 1). \quad (34)$$

It is readily verified that on putting  $y=2$  the equation (33) reduces to (24); and that on putting  $y=1$  it reduces to (25), as it should.

If on the right of (33) we substitute the first approximation

$$f(x, y) = x + \dots, \quad f(x^2, y^2) = x^2 + \dots,$$

we obtain the second approximation

$$f(x, y) = x + x^2 + \dots$$

Similarly the third approximation is

$$\begin{aligned} f(x, y) &= x + \frac{1}{2}y(x^2 + 2x^3 + x^4 + \dots) + (1 - \frac{1}{2}y)(x^2 + x^4 + \dots) \\ &= x + x^2 + x^4 + x^3y + \dots; \end{aligned}$$

and the process may be repeated to any required extent.

Alternatively, we may proceed in either of the following ways. Write

$$f(x, y) = xf_1(y) + x^2f_2(y) + \dots + x^\delta f_\delta(y) + \dots \quad (35)$$

or

$$f(x, y) = g_0(x) + yg_1(x) + y^2g_2(x) + \dots + y^\mu g_\mu(x) + \dots; \quad (36)$$

substitute in the functional equation (33), and equate coefficients. We obtain

$$\left. \begin{aligned} f_1 &= f_2 = 1, & f_3 &= y, & f_4 &= 1 + y^2, \\ f_5 &= y + y^2 + y^3, & f_6 &= y + 2y^2 + 2y^3 + y^4, & \dots, \\ f_{2\delta-1} &= y(f_1f_{2\delta-2} + f_2f_{2\delta-3} + \dots + f_{\delta-1}f_\delta), \\ f_{2\delta} &= y(f_1f_{2\delta-1} + f_2f_{2\delta-2} + \dots + f_{\delta-1}f_{\delta+1} + \frac{1}{2}f_\delta^2) + (1 - \frac{1}{2}y)f_\delta(y^2); \end{aligned} \right\} \quad (37)$$

$$\left. \begin{aligned} g_0 &= x(1-x)^{-1}, & g_1 &= x^3(1-x)^{-1}(1-x^2)^{-1}, \\ g_2 &= x^4(1+x+2x^2)(1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}, \\ g_3 &= x^5(1+x+3x^2)(1-x)^{-2}(1-x^2)^{-1}(1-x^4)^{-1}, & \dots, \\ g_{2\mu-1} &= g_0g_{2\mu-2} + g_1g_{2\mu-3} + \dots + g_{\mu-2}g_\mu + \frac{1}{2}g_{\mu-1}^2 - \frac{1}{2}g_{\mu-1}(x^2), \\ g_{2\mu} &= g_0g_{2\mu-1} + g_1g_{2\mu-2} + \dots + g_{\mu-1}g_\mu + g_\mu(x^2). \end{aligned} \right\} \quad (38)$$

It will be observed that

$$f_\delta(2) = a_\delta, \quad f_\delta(1) = b_\delta. \quad (39)$$



The first of these two methods is perhaps the quickest way of calculating many terms of the expansion of  $f(x, y)$ . By means of the second, we could find explicit formulæ for  $c_{80}, c_{81}, c_{82}, c_{83}, \dots$

With regard to the convergence of the various generating series, it may be observed that (22), since it is the expansion of (26), is absolutely convergent if  $|x| < \frac{1}{4}$ . Since  $b_8 \leq a_8$ , it follows that (23) also converges absolutely if  $|x| < \frac{1}{4}$ ; and since  $f(x, 2) = F(x)$ , it follows that the double series (32) converges absolutely if  $|x| < \frac{1}{4}, |y| \leq 2$ .

#### SUMMARY.

Non-associative combinations are classified and enumerated with the aid of a representation involving non-associative arithmetic.

#### REFERENCES TO LITERATURE.

- BINET, M. J., 1839. "Réflexions sur le problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyens de ses diagonales," *Journ. de Math.* (1), vol. iv, pp. 79-90.
- CATALAN, E., 1838. "Note sur une équation aux différences finies," *Journ. de Math.*, (1) vol. iii, pp. 508-516. Also 1839, vol. iv, pp. 91-94, 95-99; 1841, vol. vi, p. 74.
- CAYLEY, A., 1857, 1859. "On the theory of the analytical forms called trees," *Phil. Mag.*, vol. xiii, pp. 172-176; vol. xviii, pp. 374-378. Also 1875, *Rep. Brit. Assoc. Adv. Sci.*, pp. 257-305; 1881, *Amer. Journ. Math.*, vol. iv, pp. 266-268; 1889, *Quart. Journ. Pure App. Math.*, vol. xxiii, pp. 376-378. (*Collected Math. Papers*, vol. iii, no. 203; iv, 247; ix, 610; xi, 772; xiii, 895.)
- ETHERINGTON, I. M. H., 1937. "Non-associate powers and a functional equation," *Math. Gaz.*, vol. xxi, pp. 36-39, 153.
- NETTO, E., 1901. *Lehrbuch der Combinatorik*, Leipzig.
- RODRIGUES, O., 1838. "Sur le nombre de manières d'effectuer un produit de  $n$  facteurs," *Journ. de Math.* (1), vol. iii, p. 549.
- SCHRÖDER, E., 1870. "Vier combinatorische Probleme," *Zeits. Math.*, vol. xv, pp. 361-376.
- WEDDERBURN, J. H. M., 1922. "The functional equation  $g(x^2) = 2ax + [g(x)]^2$ ," *Ann. Math.* (2), vol. xxiv, pp. 121-140.

(Issued separately June 27, 1939.)

I.

Extracted from the  
*Edinburgh Mathematical Notes*, No. 32, of the  
*Edinburgh Mathematical Society*.

### Some problems of non-associative combinations (I)

By I. M. H. ETHERINGTON.

The problems considered here are essentially algebraic; but it is convenient to begin with a picturesque formulation.

Let a convex polygon cut out of paper be cut along a diagonal; it is thus divided into two convex polygons... Either of these may then again be cut along a diagonal making three convex polygons; and the process may be continued until only triangles are left, or terminated earlier, as desired. When  $r$  cuts have been made, the original polygon has been dissected into  $r + 1$  sub-polygons.

Such a dissection will be called a *partition* of the polygon. Geometrically, a partition may be described as a set of  $r$  diagonals which do not intersect in the interior. The polygon itself ( $r = 0$ ) is included among its partitions. We may enquire in how many ways a partition can be made for a given polygon, with perhaps some restriction on the kinds of sub-polygon (triangles, quadrilaterals, etc.) which may be left.

This is essentially a problem of *non-associative combinations*<sup>1</sup>, or combinations with brackets inserted; for it will be shown that if the given polygon has  $n + 1$  sides, the partitions may be described algebraically as the different ways of inserting brackets in a product  $\alpha\beta\gamma \dots$  of  $n$  factors in a given order; which will be called *associations* of the  $n$  factors.

To see this, select a particular side of the given polygon as *base*, and let the other sides be labelled  $\alpha, \beta, \gamma, \dots$ . Since the order in which the cuts are performed is not taken into account, we may proceed as follows. Regard the given polygon as an elastic band stretched tightly round  $n + 1$  pins at its vertices, the two base pins being kept fixed and the others removed in  $r$  stages, corresponding to the  $r$  cuts in a suitable order. At each stage two or more sides  $\lambda, \mu, \dots$  collapse on to a new side, previously a diagonal: if this new side is labelled as a product  $\lambda\mu \dots$ , then ultimately the base itself will be labelled as a non-associative product containing as factors the  $n$  sides  $\alpha, \beta, \gamma, \dots$  of the original polygon in order, the manner in which they are associated being determined by the partition. (See the example in figure 1.)

<sup>1</sup> Of a more general kind than those considered in my paper under this title, *Proc. Roy. Soc., Edin.*, 59, 1939, 153-162.

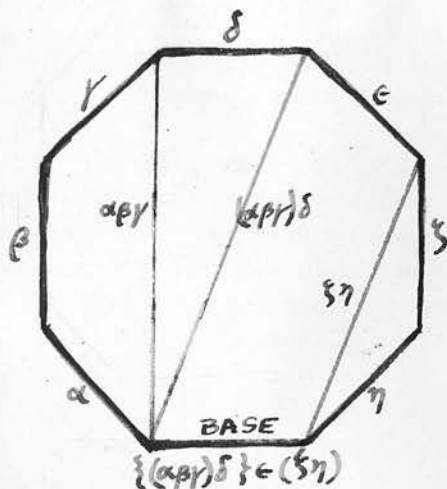


Figure 1

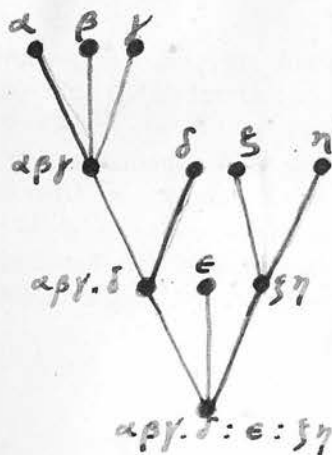


Figure 2

Conversely, any manner of inserting  $r$  pairs of brackets in the product  $\alpha\beta\gamma\dots$  corresponds to one definite partition of the given polygon with chosen base by  $r$  cuts, if the following points are observed:

- (i) The brackets must be effective; i.e., unnecessary brackets as in  $(\alpha\beta\gamma\dots)$  enclosing the complete product,  $\alpha(\beta\gamma)\dots$ , or  $\alpha((\beta\gamma))\delta\dots$ , are not to be counted. They would be like cutting the polygon along the base, along a side, or twice along the same diagonal.
- (ii) Pairs of brackets must not overlap as in  $\alpha(\beta[\gamma]\delta)$ ; for this would determine a set of diagonals intersecting in the interior of the polygon.

We have thus set up a one-one correspondence between all possible partitions of a convex polygon with  $n+1$  sides, and all possible associations of  $n$  similar objects; or, what comes to the same thing, of  $n$  dissimilar objects in a prescribed order. Combinations of this kind were represented by Cayley<sup>1</sup> as *trees*. Figure 2 will make this representation clear without further explanation. For the present purpose, corresponding to the restriction (i) above, we must consider only trees which at every knot bifurcate at least. (In some of Cayley's investigations, knots such as  $\frac{1}{2}$  were permitted.) There is also exhibited in figure 2 a convenient notation for dispensing with brackets when writing non-associative products. Dots are placed between the grouped factors, more dots implying more delay in combination.

<sup>1</sup> E.g., *Phil. Mag.*, 13 (1857), 172-176.

Consider now various special cases of the problem. In each case the required number is denoted by  $A_n$ . Taking

$$A_1 = 1,$$

the enumeration is made with the help of the generating function

$$f(x) = \sum A_n x^n \quad (n = 1, 2, 3, \dots, \infty).$$

Case 1 is equivalent to the first, and Case 5 to the second, of Schröder's *Vier combinatorische Probleme*<sup>1</sup>. Case 1 was discussed in a series of papers by Lamé, Catalan, Rodrigues, Binet<sup>2</sup>, and has been touched on by Cayley<sup>3</sup>, Wedderburn<sup>4</sup>, Etherington<sup>5</sup>. The other cases as far as I know are new. The solution of the general problem, Case 4, is completed in the next Note. The connection between the geometrical and algebraical problems was noticed by Catalan for Case 1 only.

*Case 1. Partition into triangles.*

In ~~my~~ paper III (1939, ~~loc. cit.~~) I confined attention to non-associative products in which factors are combined only two at a time. The manner of association of the factors was called the *shape* of the product. In the case when multiplication is non-commutative as well as non-associative, shapes form a special class of the associations considered above. They correspond to partitions of a convex polygon into triangles. They also correspond to trees bifurcating at every knot, which I called *pedigrees*.

The enumeration in this case is given by the following formulae. Considering how the product of  $n$  factors may be built up,

$$\begin{aligned} A_n &= A_1 A_{n-1} + A_2 A_{n-2} + \dots + A_{n-1} A_1 \quad (n > 1) \\ &= \text{the coefficient of } x^n \text{ in } f(x)^2; \end{aligned}$$

whence

$$f(x) = x + f(x)^2,$$

i.e.,

$$f(x)^2 - f(x) + x = 0.$$

<sup>1</sup> *Zeits. Math.*, 15 (1870), 361-376.

<sup>2</sup> *Journ. de Math.*, 3, 4 (1838-39), 505, etc.

<sup>3</sup> *Phil. Mag.*, 18 (1859), 374-378.

<sup>4</sup> *Ann. Math.* (2), 24 (1922), 121-140.

<sup>5</sup> *Loc. cit.*, and *Math. Gaz.*, 21 (1937), 36-39. = *Papers*, II & I

This, with  $f(0) = 0$ , gives

$$f(x) = \frac{1}{2} \{1 - (1 - 4x)^{\frac{1}{2}}\},$$

yielding on expansion

$$\begin{aligned} A_n &= \frac{(2n-2)!}{(n-1)! n!} = \frac{1}{n} \binom{2n-2}{n-1} \\ &= 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \dots \text{ for } \\ n &= 1, \dots, 10, \dots \end{aligned}$$

*Case 2. Partition into quadrilaterals.*

This corresponds to trees which trifurcate at each knot; and to the associations formed by combining factors always three at a time, e.g.,  $\alpha\beta\gamma:\delta.\epsilon\zeta\eta.\theta:\iota$ . For this to be possible, the total number of factors must be *odd*, since at each combining operation the number of factors left is reduced by two. Similarly a polygon to be partitioned into quadrilaterals must have an *even* number of sides. Thus

$$A_2 = A_4 = A_6 = \dots = 0.$$

Proceeding as before,

$$\begin{aligned} A_n &= \sum A_i A_j A_k \quad (i+j+k=n, n>1) \\ &= \text{the coefficient of } x^n \text{ in } f(x)^3; \quad (n>1) \\ f(x) &= x + f(x)^3; \end{aligned}$$

and so the generating function is that root of the cubic equation

$$f(x)^3 - f(x) + x = 0$$

for which  $f(0) = 0$ . To calculate the values of  $A_1, A_3, A_5, \dots$  in succession, we may use the method of successive approximations:

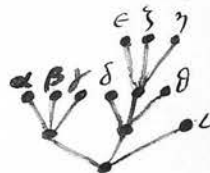
$$\begin{aligned} f(x) &= x + f(x)^3 &&= x + (x + \dots)^3 \\ &= x + x^3 + \dots &&= x + (x + x^3 + \dots)^3 \\ &= x + x^3 + 3x^5 + \dots &&= x + (x + x^3 + 3x^5 + \dots)^3, \text{ etc.} \end{aligned}$$

Continuing, it will be found that for  $n = 1, 3, 5, \dots, 19, \dots$

$$A_n = 1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, \dots$$

*Case 3. Partition into triangles or quadrilaterals.*

This corresponds to trees which divide into either two or three branches at each knot; and to associations formed by combining





factors either two or three at a time. (E.g., see the figures.) We have, for  $n > 1$ ,

$$A_n = \sum A_i A_j + \sum A_k A_l A_m \quad (i+j=k+l+m=n)$$

= the coefficient of  $x^n$  in  $f(x)^2 + f(x)^3$ .

Therefore  $f(x) = x + f(x)^2 + f(x)^3$ ,

and the generating function is that root of this equation for which  $f(0) = 0$ . By successive approximations,

$$f(x) = x + (x + \dots)^2 + (x + \dots)^3 = x + x^2 + \dots$$

$$= x + (x + x^2 + \dots)^2 + (x + x^2 + \dots)^3 = x + x^2 + 3x^3 + \dots, \text{ etc.};$$

and we find that for  $n = 1, 2, \dots, 10, \dots$

$$A_n = 1, 1, 3, 10, 38, 154, 645, 2853, 12844, 58985, \dots$$

*Case 4. Generalisation.*

Suppose we wish to enumerate the partitions of a convex  $(n+1)$ -gon with the restriction imposed that the final sub-polygons shall be all either  $(a+1)$ -gons or  $(b+1)$ -gons or  $(c+1)$ -gons, etc., where  $a, b, c, \dots$  are given positive integers. Following the method of previous cases, we arrive at the result that the generating function  $f(x)$  is determined by

$$f^a + f^b + f^c + \dots - f + x = 0, \quad f(0) = 0.$$

An explicit expression for  $A_n$  is given in formula (2) of the Note which follows.

*Case 5. The unrestricted problem.*

If no restrictions are imposed on the partitions,

$$f = x + f^2 + f^3 + f^4 + \dots \text{ to } \infty$$

$$= x + f^2/(1-f).$$

Hence  $2f^2 - (1+x)f + x = 0$ .

This agrees with Schröder's result, found in a more complicated way. He deduced

$$f(x) = \frac{1}{4} \{1 + x - (1 - 6x + x^2)^{\frac{1}{2}}\},$$

and hence

$$A_n = \frac{1}{4} (-1)^{n-1} \sum \binom{\frac{1}{2}}{n-a} \binom{n-a}{a} 6^{n-2a}$$

where  $a = 0, 1, 2, \dots; 2a \leq n; n > 1$ .

For  $n = 1, 2, 3, \dots, 10, \dots$  this gives the sequence

$$A_n = 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 41128, \dots$$

## Some problems of non-associative combinations (2)

By A. ERDÉLYI and I. M. H. ETHERINGTON.

§ 1. The preceding Note has shown the connection between the partition of a convex polygon by non-crossing diagonals and the insertion of brackets in a product, the latter being more commonly represented by the construction of a tree. It was shown that the enumeration of these entities leads to a generating function  $y = f(x)$  which satisfies an algebraic equation of the type

$$y = x + y^a + y^b + \dots \quad (a, b, \dots > 1). \quad (1)$$

In simple cases, the solution of the equation was found as a power series in  $x$ , the coefficient  $A_n$  of  $x^n$  giving the required number of partitions of an  $(n+1)$ -gon.

Now it has been shown by several writers<sup>1</sup> that every root of an algebraic equation can be expressed as a hypergeometric function of the coefficients of the equation. The generating function must thus be expressible as a hypergeometric function of  $x$ , from which an explicit formula for  $A_n$  might be deduced. The formulae obtained by these writers, however, are not immediately applicable here, because the form of the equation generally assumed differs from (1). It is of course possible to transform (1) into the usual form; but then the application of the known formulae leads only to an expansion of  $y$  in *descending* powers of  $x$ . Instead of converting this into a series of *ascending* powers as required for our problem, we have found it simpler to attack the equation afresh. As a matter of fact, our work is closely related to Birkeland's method, the device used being equivalent to his application of Lagrange's inversion formula<sup>2</sup>.

We shall show in §§ 2, 3 that the coefficient of  $x^n$  in the generating function which is a solution of (1) is

$$\left. \begin{aligned} \sum_{\alpha, \beta, \dots} \frac{(n + \alpha + \beta + \dots - 1)!}{n! \alpha! \beta! \dots} \\ \text{where } (a-1)\alpha + (b-1)\beta + \dots = n-1. \end{aligned} \right\} \quad (2)$$

<sup>1</sup> See e.g. Birkeland, several notes in *Comptes rendus*, 171 (1920) and 172 (1921); Belardinelli, *Annali di Mat.* (3), 29 (1920), 251, *Rend. de Lincei* (5), 30 (1920), 208, *Rend. di Palermo*, 46 (1922), 463; Mayr, *Monatshefte für Math. und Phys.*, 45 (1937), 280 and 47 (1938), 164; Mellin, *Ann. Acad. Sci. Fenn.* (A) 7 (1914-15), Nos. 7 and 8.

<sup>2</sup> Cf. Whittaker and Watson, *Modern Analysis* (1927), § 7.32.

This therefore represents the number of partitions of an  $(n+1)$ -gon into  $(a+1)$ -gons,  $(b+1)$ -gons, .... Having obtained this result, it was natural to look for an interpretation of each term in this summation. This led us to consider a more specialised problem, that of enumerating the partitions of an  $(a+1)$ -gon into  $a$   $(a+1)$ -gons,  $\beta$   $(b+1)$ -gons, ...., where  $n, a, \alpha, \beta, b, \dots$  are all given numbers. Beginning therefore with this problem we shall establish a more fundamental formula (8), from which (2) follows as an easy corollary.

As in the preceding Note, the solution depends in the first instance on the derivation of a non-linear recurrence relation. Now the earlier writers<sup>1</sup> who considered the particular case of partition into triangles ( $a=2, \alpha=n-1, \beta=\dots=0$ ; Case 1 of the preceding Note) observed that the non-linear recurrence relation which occurs there is equivalent to the linear recurrence relation

$$B_n = 2(2n-3)B_{n-1} \quad (B_n = n! A_n),$$

suitable initial conditions being prescribed in both cases. This connection between a non-linear and a linear recurrence equation persists in the problem just enunciated, but seemingly not in the less detailed problem with which we started. This will be shown in §4; the argument used is a generalisation of that of Rodrigues (*loc. cit.*, p. 549).

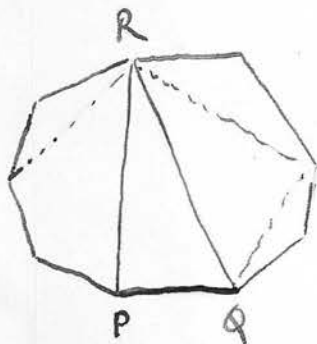
§2. We consider for simplicity a special case ( $a=2, b=3$ ) of the problem enunciated in §1. Suppose that a convex  $(n+1)$ -gon is partitioned into  $\alpha$  triangles and  $\beta$  quadrilaterals. It will appear from the solution that we must have

$$n = \alpha + 2\beta + 1; \quad (3)$$

and this may be also shown directly without difficulty. Nevertheless it is convenient to regard the number of partitions as depending on  $n, \alpha, \beta$  independently; it will be denoted  $A_{n\alpha\beta}$ , this being zero if the condition (3) is not satisfied. We define  $A_{100} = 1$ .

Let one side  $PQ$  of the polygon be selected as base. Suppose first that the part of the polygon which contains the base is a triangle  $PQR$ ; that  $PR$  is the base of an  $(n'+1)$ -gon divided into  $\alpha'$  triangles and  $\beta'$  quadrilaterals; and that  $QR$  is the base of an  $(n''+1)$ -gon divided into  $\alpha''$  triangles and  $\beta''$  quadrilaterals. We must evidently have

$$n' + n'' = n, \quad \alpha' + \alpha'' + 1 = \alpha, \quad \beta' + \beta'' = \beta, \quad (4)$$



<sup>1</sup> Lamé, Catalan, Rodrigues, Binet, *Journ. de Math.*, 1838-39.

the term “+ 1” accounting for the triangle  $PQR$ . The number of ways in which this can occur is  $A_{n' a' \beta'} A_{n'' a'' \beta''}$ ; and hence the total number of partitions of the original polygon in which  $PQ$  forms one side of a triangle is

$$\sum A_{n' a' \beta'} A_{n'' a'' \beta''},$$

the summation being over all values of the suffixes satisfying (4).

We can enumerate similarly the partitions which contain a quadrilateral  $PQRS$ ; and we arrive at the recurrence relation

$$A_{na\beta} = \sum A_{n' a' \beta'} A_{n'' a'' \beta''} + \sum A_{n' a' \beta'} A_{n'' a'' \beta''} A_{n''' a''' \beta'''} \quad (n > 1), \quad (5)$$

where in the first summation

$$n' + n'' = n, \quad a' + a'' = a - 1, \quad \beta' + \beta'' = \beta,$$

and in the second

$$n' + n'' + n''' = n, \quad a' + a'' + a''' = a, \quad \beta' + \beta'' + \beta''' = \beta - 1.$$

Introducing the generating function

$$y = f(x, A, B) = \sum A_{na\beta} x^n A^a B^\beta,$$

it follows from (5) that

$$y = x + Ay^2 + By^3.$$

By the same argument, it will be seen that the number  $A_{na\beta} \dots$  of partitions of an  $(n+1)$ -gon into  $a$   $(a+1)$ -gons,  $\beta$   $(\beta+1)$ -gons,  $\dots$  is the coefficient of  $x^n A^a B^\beta \dots$  in the expansion of  $y$ , where  $y$  is that root of the equation

$$y = x + Ay^a + By^b + \dots \quad (6)$$

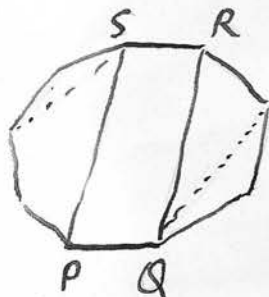
which vanishes for  $x = 0$ .

§ 3. Instead of (6), we consider the rather more general equation

$$x = y F(y)$$

where  $F(y)$  is an analytic function of the complex variable  $y$ , one-valued and regular in a certain neighbourhood of  $y = 0$ . We assume  $F(0) \neq 0$ , and may take without loss of generality  $F(0) = 1$  (otherwise replace  $x$  by  $x F(0)$  and divide the equation by  $F(0)$ ). The solution of this equation presents no greater difficulties than that of (6).

Then there is one and only one branch of the function  $y = f(x)$  which vanishes for  $x = 0$ . This branch is an analytic function of  $x$ , one-valued and regular in a certain domain of the  $x$ -plane. We



x

suppose  $x$  (and later on  $\xi$ ) to be inside this domain and  $y$  (and  $\eta$ ) to be inside the corresponding domain in the  $y$ -plane.

Now consider the integral

$$\frac{1}{2\pi i} \int_C \frac{d\eta}{\eta F(\eta) - x}$$

in which the contour  $C$  of the  $\eta$ -plane encircles  $\eta = y$  in the positive direction remaining throughout inside the domain of one-to-one correspondence between  $\eta$  and  $\xi = \eta F(\eta)$ . Introducing  $\xi$  as a new variable of integration we obtain

$$\frac{1}{2\pi i} \int_{C'} \frac{d\eta}{d\xi} d\xi / (\xi - x).$$

$C'$  is the image of  $C$  in the  $\xi$ -plane and encircles  $\xi = x$  in the positive direction.  $\eta$  and hence  $d\eta/d\xi$  being one-valued analytic functions, regular in a domain containing  $C'$  entirely, the value of the integral is

$$\left. \frac{d\eta}{d\xi} \right|_{\xi=x} = \frac{dy}{dx}.$$

On the other hand, if  $x$  is sufficiently small, we can choose a contour  $C$  which is a circle with origin  $\eta = 0$ , and in every point of which  $|x| < |\eta F(\eta)|$ . Then the expansion

$$\frac{1}{\eta F(\eta) - x} = \frac{1}{\eta F(\eta)} \left( 1 - \frac{x}{\eta F(\eta)} \right)^{-1} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{\{\eta F(\eta)\}^n};$$

is uniformly convergent on  $C$  and may be integrated term by term. Hence

$$\frac{dy}{dx} = \frac{1}{2\pi i} \int_C \frac{d\eta}{\eta F(\eta) - x} = \sum_{n=1}^{\infty} x^{n-1} \frac{1}{2\pi i} \int_C \frac{d\eta}{\eta^n F^n(\eta)}$$

and,  $f(0)$  being equal to zero,

$$y = \int_0^x \frac{dy}{dx} dx = \sum_{n=1}^{\infty} \frac{x^n}{n} \frac{1}{2\pi i} \int_C \frac{d\eta}{\eta^n F^n(\eta)}.$$

This result can be expressed as follows: the coefficient of  $x^n/n$  in the expansion of  $y$  in ascending powers of  $x$  is equal to the coefficient of  $y^{n-1}$  in the expansion of  $\{F(y)\}^{-n}$  in ascending powers of  $y$ .

Let us apply this rule to the algebraic equation (6). In this case

$$F(y) = 1 - Ay^{a-1} - By^{b-1} - \dots;$$

and hence by the multinomial theorem

$$\{F(y)\}^{-n} = \sum_{\alpha, \beta, \dots} \frac{(n + \alpha + \beta + \dots - 1)!}{(n-1)! \alpha! \beta! \dots} A^\alpha B^\beta \dots y^{(a-1)\alpha + (b-1)\beta + \dots},$$



the summation being extended over all non-negative integer values of  $\alpha, \beta, \dots$ . The coefficient of  $y^{n-1}$  in this expansion is

$$\sum_{\alpha, \beta, \dots} \frac{(n + \alpha + \beta + \dots - 1)!}{(n-1)! \alpha! \beta! \dots} A^\alpha B^\beta \dots$$

where  $(\alpha-1)\alpha + (\beta-1)\beta + \dots = n-1$ .

Hence we get by our rule

$$y = f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{\alpha, \beta, \dots} \frac{(n + \alpha + \beta + \dots - 1)!}{(n-1)! \alpha! \beta! \dots} A^\alpha B^\beta \dots \quad (7)$$

where  $(\alpha-1)\alpha + (\beta-1)\beta + \dots = n-1$  in the inner sum; that is,

$$y = \sum \frac{(\alpha\alpha + \beta\beta + \dots)!}{[(\alpha-1)\alpha + (\beta-1)\beta + \dots + 1]! \alpha! \beta! \dots} x^{(\alpha-1)\alpha + (\beta-1)\beta + \dots} A^\alpha B^\beta \dots$$

$(\alpha, \beta, \dots = 0, 1, 2, \dots).$

We have thus shown that

$$A_{n\alpha\beta\dots} = \frac{(n + \alpha + \beta + \dots - 1)!}{n! \alpha! \beta! \dots} \quad (8)$$

provided that

$$n = (\alpha-1)\alpha + (\beta-1)\beta + \dots + 1;$$

and is otherwise zero.

Returning to the original problem where  $\alpha, \beta, \dots$  are *not* specified, we have to deal with equation (1), and thus merely to put in (6)

$$A = B = \dots = 1.$$

Correspondingly (2) follows from (7); or may be deduced from (8).

§ 4. Dropping the redundant suffix  $n$  used in § 2, let  $A_{\alpha\beta}$  denote the number of partitions of a polygon with  $\alpha + 2\beta + 2$  sides into  $\alpha$  triangles and  $\beta$  quadrilaterals. This may also be interpreted as the number of associations of  $\alpha + 2\beta + 1$  identical elements in the form of a product containing  $\alpha$  couplets and  $\beta$  triplets. (A *couplet* or a *triplet* means a pair of brackets which unites two or three factors respectively.) Hence

$$B_{\alpha\beta} = (\alpha + 2\beta + 1)! A_{\alpha\beta} = \frac{(2\alpha + 3\beta)!}{\alpha! \beta!}$$

enumerates the associations of  $\alpha + 2\beta + 1$  *distinct* elements in the same form. Let us call these  $B_{\alpha\beta}$  non-associative products *the associations*  $(\alpha, \beta)$ .

Suppose that a particular element in an association  $(\alpha, \beta)$  is obliterated. The element may be one of a couplet or one of a triplet, and the association becomes accordingly either  $(\alpha-1, \beta)$  or

$(a+1, \beta-1)$ . Conversely, any association  $(a-1, \beta)$  may be reconverted into  $(a, \beta)$  by the insertion of a single element in

$$2\{1 + 2(a-1) + 3\beta\} = 2(2a + 3\beta - 1)$$

ways; for the element may be introduced as a pre- or post-multiplier of the whole association, of either factor in any of the  $a-1$  couplets or of one factor in any of the  $\beta$  triplets. Similarly any association  $(a+1, \beta-1)$  may be reconverted into  $(a, \beta)$  by inserting a single element in  $3(a+1)$  different ways; for it may be introduced in three positions into any of the  $a+1$  couplets.

Consider now all the associations  $(a, \beta)$  of  $a+2\beta+1$  given elements. When a particular element is obliterated, we are left with all the associations  $(a-1, \beta)$  each repeated  $2(2a+3\beta-1)$  times, and all the associations  $(a+1, \beta-1)$  each repeated  $3(a+1)$  times. Thus

$$B_{a,\beta} = 2(2a+3\beta-1)B_{a-1,\beta} + 3(a+1)B_{a+1,\beta-1}.$$

The argument may be generalised without difficulty. Taking in (8)

$$a=2, b=3, \dots$$

and dropping the redundant suffix  $n$ , we obtain

$$B_{a\beta\gamma\dots} = (a+2\beta+3\gamma+\dots+1)! A_{a\beta\gamma\dots} = \frac{(2a+3\beta+4\gamma+\dots)!}{a!\beta!\gamma!\dots} \quad (9)$$

as the number of associations of  $a+2\beta+3\gamma+\dots+1$  distinct elements in the form of a bracketed product containing  $a$  couplets,  $\beta$  triplets,  $\gamma$  quadruplets, ....; and it will be found that  $B_{a\beta\gamma\dots}$  satisfies the linear recurrence relation

$$B_{a,\beta,\gamma,\dots} = 2(2a+3\beta+4\gamma+\dots-1)B_{a-1,\beta,\gamma,\dots} + 3(a+1)B_{a+1,\beta-1,\gamma,\dots} + 4(\beta+1)B_{a,\beta+1,\gamma-1,\dots} + \dots,$$

this formula being easily verified by direct substitution of (9).

THE UNIVERSITY, EDINBURGH.

NOTE. Added 28th January, 1940.

Since this paper was written, a paper of G. Belardinelli (*Monatsh. Math. Phys.* 48 (1939), 381-388) has appeared which contains the general result of § 3, namely the formula

$$y = \sum_{n=1}^{\infty} \frac{x^n}{n} \cdot \frac{1}{2\pi i} \int^{(0+)} \frac{d\eta}{\eta^n F^n(\eta)}.$$

From this expression Belardinelli proves  $y$  to be a hypergeometric series of infinite order in  $x$ .

## POSTSCRIPT TO PART ONE

The classification of shapes given in Paper II, §3 can be extended to the more general non-associative combinations considered in Papers III and IV. I will illustrate this in a simple way by considering the trifurcating associations of Paper III, Case 2 - let us call them triproducts, generated by trimultiplication.

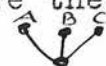
At each knot in a tree, three factors are united associatively. Degree ( $\delta$ ) and altitude ( $\alpha$ ) being defined as before, the total number of knots is  $\frac{1}{2}(\delta-1)$ , and we find easily

$$3^\alpha \geq \delta \geq 2\alpha + 1, \text{ i.e. } \frac{1}{2}(\delta-1) \geq \alpha \geq \log_3 \delta. \text{ Cf. II (14)}$$

To define mutability, let us distinguish the two cases of commutative trimultiplication, where

$$ABC = ACB = BCA = BAC = CAB = CBA,$$

and non-commutative trimultiplication, where these are all distinct.

Using the word conformal as before, a knot  which unites three factors A, B, C (themselves triproducts) is balanced if A, B, C are conformal ( $A \sim B \sim C$ ); half-balanced if only two of them are conformal (e.g.,  $A \sim B \not\sim C$ ); unbalanced if  $A \not\sim B$ ,  $B \not\sim C$ ,  $C \not\sim A$ ; not balanced if it is either half-balanced or unbalanced. Let  $\kappa, \lambda, \mu$  be the numbers of half, un-, and not balanced knots respectively in the tree representing a given triproduct, so that

$$\mu = \kappa + \lambda.$$

Then the number of conformal non-commutative shapes is

$$3^\kappa 6^\lambda = 3^{\kappa+\lambda} 2^\lambda = 2^\lambda 3^\mu.$$

This is evidently the beginning of a factorial expression analogous to the  $r!$  which connects the numbers of associative combinations and permutations of  $n$  things taken  $r$  at a time. It seems natural to define mutability here as a double quantity having the two components  $\lambda, \mu$  (we might call them lambdability and mutability).

Apart from the case of a triproduct consisting of a single factor only ( $\delta = 1$ ), one knot must necessarily be balanced, and so  $\mu$  cannot exceed  $\frac{1}{2}(\delta-1) - 1$ . Thus

$$\delta \geq 2\mu + 3, \text{ except when } \delta = 1; \text{ Cf. II (15)}$$

and no doubt relations exist analogous to II(16), (17).

It has already been shown how to enumerate the non-commutative

triproduct shapes of given degree (Paper III, Case 2). Let  $B_\delta$  be the number of possible commutative shapes of degree  $\delta$ ,  $B_\delta$  being defined as zero when  $\delta$  is not a positive integer.

The recurrence equation analogous to II (19) may be shown to be

$$B_\delta = \sum B_i B_j B_k + \frac{1}{2} \sum B_i (B_i + 1) B_m + \frac{1}{3} B_n (B_n + 1) (B_n + 2),$$

where  $\delta > 1$ ,  $i < j < k$ ,  $l \neq m$ ,  $i + j + k = 2l + m$  &  $3n = \delta$ .  
Thus

$$B_1 = 1,$$

$$B_3 = \frac{1}{3!} B_1 (B_1 + 1) (B_1 + 2) = 1,$$

$$B_5 = \frac{1}{2!} B_1 (B_1 + 1) B_3 = 1,$$

$$B_7 = \frac{1}{2!} \{ B_1 (B_1 + 1) B_5 + B_3 (B_3 + 1) B_1 \} = 2,$$

$$B_9 = B_1 B_3 B_5 + \frac{1}{2!} B_1 (B_1 + 1) B_7 + \frac{1}{3!} B_3 (B_3 + 1) (B_3 + 2) = 4.$$

The four commutative triproduct shapes of degree 9 are in fact

$$3.3, \quad 1+(1+3+1)+3, \quad 1+(3+1+3)+1, \quad 1+\{1+(1+3+1)+1\}+1$$



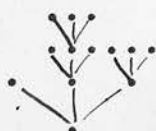
$$\alpha = 2,$$

$$\beta = 0,$$

$$\lambda = 0,$$

$$2^\alpha 3^\beta = 1.$$

(Plenary)

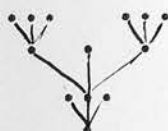


$$\alpha = 3,$$

$$\beta = 2,$$

$$\lambda = 1,$$

$$2^\alpha 3^\beta = 18.$$



$$\alpha = 3,$$

$$\beta = 2,$$

$$\lambda = 0,$$

$$2^\alpha 3^\beta = 9.$$



$$\alpha = 4,$$

$$\beta = 3,$$

$$\lambda = 0,$$

$$2^\alpha 3^\beta = 27.$$

(Primary)

~~As a check on the recurrence formula, I have counted~~

Constructing their trees and counting their mutabilities as

above, the total number of conformal non-commutative shapes is found to be

$$A_9 = \sum 2^\alpha 3^\beta = 1 + 18 + 9 + 27 = 55,$$

agreeing with the calculation in Paper III, and confirming the formulae given here.

Writing

$$f(x) = B_1 x + B_3 x^3 + B_5 x^5 + \dots,$$

the functional equation analogous to II (25) is

$$f(x)^3 + 2 f(x^3) + 3 f(x) f(x') - 6 f(x) + 6x = 0.$$

An extension of this theory could be developed in connection with the "Algebras of  $s$  dimensions" considered by A.R. Richardson, Proc. Lond. Math. Soc., (2) 47 (1940) 38-59.

PART TWO

THE SYMBOLISM OF GENETICS



## PROCEEDINGS

OF THE

## ROYAL SOCIETY OF EDINBURGH

Section B (Biology)

VOL. LXI—PART I (No. 2)

24

## II.—Non-Associative Algebra and the Symbolism of Genetics.

By I. M. H. Etherington, B.A., Ph.D., Mathematical Institute,  
University of Edinburgh.

(MS. received August 14, 1940. Read December 2, 1940.)

## CONTENTS.

|  | PAGE |  | PAGE |
|--|------|--|------|
| § 1. INTRODUCTION . . . . .                                  | 24   | § 7. THE MENDELIAN COPULAR                                 |      |
| § 2. GENETICAL MULTIPLICA-<br>TION . . . . .                 | 25   | ALGEBRA . . . . .  | 32   |
| § 3. NON-ASSOCIATIVE PRODUCTS<br>AND POWERS . . . . .        | 26   | § 8. SYSTEMS OF MATING . . . . .                           | 34   |
| § 4. LINEAR ALGEBRAS . . . . .                               | 28   | § 9. COMPACT MULTIPLICATION<br>TABLES . . . . .            | 37   |
| § 5. THE MENDELIAN GAMETIC<br>AND ZYGOTIC ALGEBRAS . . . . . | 29   | § 10. OFFSPRING OF CONSAN-<br>GUINEOUS MARRIAGES . . . . . | 38   |
| § 6. "SHORTCIRCUITED" MUL-<br>TIPLICATION . . . . .          | 31   | § 11. FURTHER GENETIC ALGEBRAS                             | 39   |
|  |      | SUMMARY . . . . .  | 41   |
|  |      | REFERENCES TO LITERATURE                                   | 42   |

## § 1. INTRODUCTION.

THE statistical material of genetics usually consists of frequency distributions—of genes, zygotes and mating couples—from which new distributions referring to their progeny arise. Combination of distributions by random mating is usually symbolised by the mathematical sign for multiplication; but this sign is not taken literally for the simple reason that the genetical laws connecting the distributions of progenitors and progeny are inconsistent with the laws governing multiplication in ordinary algebra. This is explained more fully in § 2.

However, there is no insuperable reason why the genetical sign of multiplication should not be taken literally; for it is possible with any particular type of inheritance to construct an "algebra"—distinct from ordinary algebra but of a type well known to mathematicians—such that the laws governing multiplication shall represent exactly the underlying genetical situation. These "genetic algebras" are of a kind known as "linear algebras," of which a simple description is given in § 4.

It is not suggested that the use of ordinary algebraic methods in conjunction with the specific principles of genetics will not lead to correct results. It seems, however, that the systematic use of genetic algebras would simplify and shorten the way to their attainment, and perhaps enable much more difficult problems to be tackled with equal ease.

The construction of genetic algebras has been described in a somewhat abstract way in a previous paper (Etherington, 1939 *b*), to which I shall refer as G.A. Here I propose to consider the symbolism more from the geneticist's point of view, applying it to some simple population problems, without going into the details of the mathematical background. It will be recognised that the current treatment of such problems does in reality make use of genetic algebras without noticing them explicitly. By elaborating the symbolism and adapting it to more complicated genetical premises (*e.g.* in the manner indicated in G.A. § 14), it should be possible to avoid the laborious complexity which other methods in such cases would involve.

Only elementary mathematical knowledge is assumed, and it is hoped that this paper will be found understandable by geneticists whose mathematical knowledge is quite limited.

## § 2. GENETICAL MULTIPLICATION.

Capital letters will be used to represent frequency or probability distributions, referring to either a population, a single individual, or a single gamete; such as (in the case of autosomal allelomorphs)

$P = DD$  = homozygous dominant individual, or population consisting of such;

$P = \alpha DD + \beta DR + \gamma RR$  = population with assigned frequencies  $\alpha : \beta : \gamma$  of genotypes, or individual with assigned probabilities  $\alpha, \beta, \gamma$  of belonging to one or other genotype;

$P = \delta D + \rho R$  = population which produces D and R gametes in given numerical ratio, or gamete which has probability  $\delta$  of containing D,  $\rho$  of containing R.

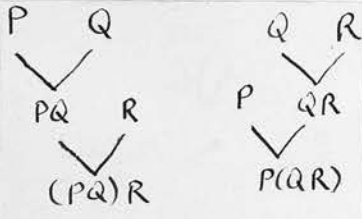
The *multiplication* of populations—individuals—gametes—means the calculation of progeny distribution resulting from their random mating—mating—fusion. Defining a *population* as a probability distribution of genetic types, we may say in all cases that we are multiplying populations.

Now multiplication in ordinary algebra obeys three laws: (1) the commutative law  $PQ = QP$ , (2) the associative law  $P(QR) = (PQ)R$ , (3) the distributive law  $P(Q + R) = PQ + PR$ .

The validity of the distributive law in the genetic symbolism is sufficiently obvious; it forms the basis of the method of "chess-board diagrams" often used as visual aids in the calculation of progeny distributions.

The associative law is not obeyed in genetical multiplication. This

(*i.e.* Paper VI)



is seen by comparing the progeny of a mating between the offspring from two individuals or populations, denoted as  $PQ$ , and a third individual or population  $R$  (*i.e.* the product  $(PQ)R$ ), with the progeny from  $P$  and the hybrid population  $QR$  (*i.e.* the product  $P(QR)$ ). There is clearly no reason why they should be the same, and in fact unless  $P=R$  they are found to be different. Thus genetical multiplication is non-associative.

Regarding the commutative law, (i) if we are considering autosomal characters it will be obvious that this law applies, since the results of reciprocal matings are generally speaking identical, although we shall see below that in certain cases non-commutative multiplication can occur.

(ii) One might be tempted to say that with sex-linked characters multiplication is non-commutative, since the results of reciprocal matings are different. But it must be remembered that with sex-linked characters we can only speak of reciprocal matings in connection with the phenotype classification of a population; whereas the calculation of progeny distribution is only possible on the basis of the genotype classification. A given genotype (either involving the Y-chromosome or not) is either female or male, so that a reciprocal mating between genotypes is impossible. Suppose that we are multiplying a male genotype  $M$  and a female genotype  $F$ : then  $MF$  and  $FM$  both mean the same thing—the genotype distribution of their offspring; and so multiplication is commutative.

(iii) On the other hand, returning to autosomal inheritance, it is possible for this to be unsymmetrical in the sexes, through either crossing-over values or gametic selection being different in male and female. In such cases it is really optional whether we treat corresponding male and female genotypes as the same type (since their relevant gene content is the same) or as distinct types (since they produce different series of gametes). In the former case,  $PQ$  and  $QP$  have distinct meanings, referring to reciprocal crosses which do not produce similar distributions of offspring; and multiplication is non-commutative. In the latter case, the situation is as with sex-linkage.

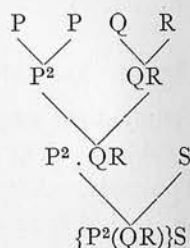
To sum up, genetical multiplication is non-associative, but obeys the commutative and distributive laws; except that in certain cases we have the option of using a varied form of the symbolism in which the multiplication is non-commutative as well as non-associative.

### § 3. NON-ASSOCIATIVE PRODUCTS AND POWERS.

Non-commutative algebra of a special kind (matrix algebra) is widely familiar by reason of its many applications in geometry and physics. (Also in genetics: *cf.* Hogben, 1933; Geppert and Koller, 1938, Chap. 4.)

Hence there is no reason to fear that an algebra which does not obey all the usual laws will necessarily prove unmanageable.

But with non-associative algebra some precautions are required to avoid confusion, especially when dealing with products or powers involving many factors. With such an expression, brackets inserted in different ways would indicate different orders of association of the factors; and the corresponding interpretations of the whole product would refer to the various pedigrees which could be constructed with given ancestors. For example, the product  $\{P^2(QR)\}S$  represents the pedigree below. The



separate factors or ancestors  $P, P, Q, R, S$  may be thought of as given genotypes or distributions of genotypes. The factor  $P^2$  may arise through self-fertilisation of an individual  $P$ , or from the mating of two individuals of the same genetic type, or from random mating within one population  $P$  or between two similar populations. The partial products  $P^2, QR, P^2(QR)$ , and the final result  $\{P^2(QR)\}S$  are probability distributions which, for any particular type of inheritance, can be calculated when  $P, Q, R, S$  are known.

To avoid clumsiness of notation, it is convenient to use groups of dots in place of brackets, fewness of dots between factors conferring precedence in multiplication. Thus the above product would be written  $P^2 \cdot QR : S$ . On putting  $P=Q=R=S$ , it becomes a power of  $P$ . (I have discussed elsewhere a notation and nomenclature for non-associative powers (1939 *a*, - *Paper II* § 2); *e.g.* the power in question is denoted  $P^{2.2+1}$ . We shall be concerned, however, with only two simple types of non-associative powers, namely, the "principal" and "plenary" powers described below.)

Similarly, the product appearing at (10.1) below denotes

$$[\{(ab)(cd)\}(ef)][\{(ab)(cd)\}(gh)].$$

The pedigree for this is easily constructed; but it should be noted that in the context  $a, b, c, \dots$  denote gametes, so that  $ab, cd, \dots$  are the ancestral zygotes.

We shall find it important to distinguish between, *e.g.*,  $P^{2.2}=(P^2)^2$  and  $P^4=P\{P(P^2)\}=P:P \cdot PP$ . If mating takes place at random in a popula-



tion, initially  $P$ , the successive generations, supposed discrete, are represented by the sequence of *plenary powers*

$$P, P^2, P^{2^2}, P^{2^3}, \dots, P^{2^{n-1}}, \dots, \quad (3.1)$$

each the square of the preceding; while the sequence of *principal powers*

$$P, P^2, P^3, \dots, P^n, \dots, \quad (3.2)$$

each obtained from the preceding by multiplication with  $P$ , refers similarly to a mating system in which each generation is mated back to one original ancestor or ancestral population.

#### § 4. LINEAR ALGEBRAS.

Linear algebras have been studied for some ninety years, and there is an extensive literature of the subject. The following brief description will be sufficient for the present purpose. Attention is confined to algebras "over the field of real numbers"; that is to say, the Greek letters below denote ordinary real numbers, and this convention will be observed throughout the paper.

Beginning with a simple case, a *commutative*\* *linear algebra of order 2* is determined when two given symbols or *units*  $A, B$  are subject to a multiplication table consisting of product rules of the form

$$A^2 = \alpha A + \beta B, \quad AB = \gamma A + \delta B, \quad B^2 = \epsilon A + \zeta B, \quad (4.1)$$

the coefficients being given numerical constants. The algebra then consists of all possible expressions of the form

$$P = \lambda A + \mu B, \quad (4.2)$$

which are called *hypercomplex numbers*.† Addition and multiplication of hypercomplex numbers are carried out as in ordinary algebra, the multiplication table (4.1) being used to reduce a product to the "linear" form (4.2). Thus if

$$P = \lambda A + \mu B, \quad Q = \nu A + \rho B,$$

then

$$P \pm Q = (\lambda \pm \nu)A + (\mu \pm \rho)B, \quad (4.3)$$

$$\left. \begin{aligned} PQ &= \lambda\nu A^2 + (\lambda\rho + \mu\nu)AB + \mu\rho B^2 \\ &= \lambda\nu(\alpha A + \beta B) + (\lambda\rho + \mu\nu)(\gamma A + \delta B) + \mu\rho(\epsilon A + \zeta B) \\ &= (\lambda\nu\alpha + \lambda\rho\gamma + \mu\nu\gamma + \mu\rho\epsilon)A + (\lambda\nu\beta + \lambda\rho\delta + \mu\nu\delta + \mu\rho\zeta)B. \end{aligned} \right\} \quad (4.4)$$

\* *Commutative* refers to the nature of multiplication; the *order* is the number of units on which the algebra is based.

† So called because they are a generalisation of the more familiar *complex numbers*. The algebra of complex numbers possesses a *real unit* 1 and an *imaginary unit*  $i$ , which are subject to the multiplication table  $1^2 = 1$ ,  $1i = i$ ,  $i^2 = -1$ .



It was implied in (4.1) that  $BA=AB$ . The linear algebra would be *non-commutative*, however, if different formulæ were prescribed for  $AB$  and  $BA$ ; and then we should have  $PQ \neq QP$ . Unless special conditions are satisfied by the coefficients in (4.1), multiplication is non-associative.

A linear algebra of order  $n$  is defined in an analogous way. It will be based on  $n$  units, and will consist of hypercomplex numbers: a hypercomplex number is an expression which is linear (*i.e.* of the first degree throughout) in the  $n$  units; and the algebra will have a multiplication table giving a linear formula for the square of each unit and for the product of each pair of units. (See, for example, the multiplication tables (5.3) and (11.10), which determine linear algebras of orders 3 and 5 respectively.)

The commutative and associative laws of addition,

$$P+Q=Q+P, \quad (P+Q)+R=P+(Q+R),$$

always hold; so do the distributive laws

$$P(Q+R)=PQ+PR, \quad (Q+R)P=QP+RP;$$

but multiplication may be non-commutative, non-associative, or both.

It will be seen that a linear algebra is completely determined when its multiplication table is known.

Given any two linear algebras of orders  $m$  and  $n$  (*i.e.* given their multiplication tables), it is possible by combining their multiplication tables in a certain way to deduce another linear algebra, of order  $mn$ , which is known as their *direct product*. This is of fundamental importance in the general theory of linear algebras, and we shall find (§ 11; *cf.* G.A. § 9) that it is also fundamental in the symbolism of genetics. If the units on which the first algebra is based are  $A, B, \dots$ , and those of the second  $A', B', \dots$ , then the units of the direct product may be interpreted as  $AA', AB', BA', BB', \dots$

Also (of less importance in the mathematical theory, but equally fundamental in genetics), from any linear algebra of order  $n$  a closely related linear algebra called its *duplicate* can be derived, of order  $\frac{1}{2}n(n+1)$  if the original algebra is commutative. If the original units are  $A, B, \dots$ , those of the duplicate algebra may be interpreted as  $A^2, B^2, AB, \dots$  (The process of duplication was described in G.A. § 5; *cf.* also Etherington, 1941; it occurs here in §§ 5-7.)

(Paper VII)

#### § 5. THE MENDELIAN GAMETIC AND ZYGOTIC ALGEBRAS.

Consider a pair of autosomal allelomorphs  $D, R$  and the corresponding genotypes

$$A=DD, \quad B=DR, \quad C=RR. \quad \dots \quad (5.1)$$

We shall write optionally DD or  $D^2$ , RR or  $R^2$ . In accordance with mendelian principles and with the notation described at the beginning of § 2, we have the two sets of formulæ:

$$D^2 = D, \quad DR = \frac{1}{2}D + \frac{1}{2}R, \quad R^2 = R; \quad (5.2)$$

$$\left. \begin{array}{lll} A^2 = A, & B^2 = \frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C, & C^2 = C, \\ BC = \frac{1}{2}B + \frac{1}{2}C, & CA = B, & AB = \frac{1}{2}A + \frac{1}{2}B. \end{array} \right\} \quad (5.3)$$

These give the series of gametes produced by each type of zygote, and the series of zygotes produced by each type of mating couple, with coefficients denoting relative frequencies. *E.g.*, the second of equations (5.2) mean that a heterozygote produces D and R gametes in equal numbers; the second of equations (5.3) means that the offspring of a mating  $DR \times DR$  are 25 per cent. DD, 50 per cent. DR, 25 per cent. RR.

A population P can be described by the frequencies either of the gametes which it produces, or of the zygotes which it contains, and accordingly we write:

$$(Gametic\ representation) \quad P = \delta D + \rho R, \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

$$(Zygotic\ representation) \quad P = \alpha A + \beta B + \gamma C \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

$$= \alpha DD + \beta DR + \gamma RR, \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

in which we may assume

$$(Normalising\ conditions) \quad \delta + \rho = 1, \quad \alpha + \beta + \gamma = 1. \quad . \quad . \quad . \quad . \quad (5.7, 8)$$

The two representations are connected by (5.2); *i.e.* (5.6) implies (5.4) with

$$\delta = \alpha + \frac{1}{2}\beta, \quad \rho = \frac{1}{2}\beta + \gamma. \quad . \quad . \quad . \quad . \quad (5.9)$$

An examination of the above formulæ in the light of § 4 will show that by using this symbolism we are really dealing with two distinct linear algebras, both having commutative and non-associative multiplication, namely:

(1) The algebra of the symbols D, R with multiplication table (5.2). This will be called the *gametic algebra for simple mendelian inheritance*, and referred to as **G**. A hypercomplex number in this algebra has the form (5.4).

(2) The algebra of the symbols A, B, C with multiplication table (5.3). Call this the *zygotic algebra for simple mendelian inheritance*, and denote it **Z**. A hypercomplex number in **Z** has the form (5.5). Hypercomplex numbers in **G** and **Z** are interpreted as populations only if their coefficients are all positive; and it is generally convenient to require that the coefficients shall satisfy the normalising conditions (5.7, 8).

The relation between the two algebras is given by (5.1), which means that a hypercomplex number (or *linear form*) (5.5) in  $\mathbf{Z}$  is equivalent to a *quadratic form* (5.6) in  $\mathbf{G}$ . The quadratic form is reduced to a hypercomplex number in  $\mathbf{G}$  by using the multiplication table (5.2). That is to say (*cf.* 5.9), the zygotic representation determines the gametic; but not *vice versa*, owing to the extra degree of freedom in the zygotic algebra.

Starting from the gametic multiplication table (5.2), the equations (5.3) are built up by the following process: we take the symbols A, B, C defined by (5.1) as units of a new algebra, and then

$$\left. \begin{aligned} A^2 &= DD, DD = D, D = A, \\ B^2 &= (\tfrac{1}{2}D + \tfrac{1}{2}R)^2 = \tfrac{1}{4}DD + \tfrac{1}{2}DR + \tfrac{1}{4}RR = \tfrac{1}{4}A + \tfrac{1}{2}B + \tfrac{1}{4}C, \\ AB &= DD, DR = D(\tfrac{1}{2}D + \tfrac{1}{2}R) = \tfrac{1}{2}A + \tfrac{1}{2}B, \end{aligned} \right\} \quad (5.10)$$

and so on. Thus the zygotic multiplication table is constructed from the gametic. This is the process of duplication referred to in § 4, and  $\mathbf{Z}$  is thus the duplicate of  $\mathbf{G}$ .

Suppose that we wish to find the progeny distribution of two mating populations P, Q, whose representations, either gametic or zygotic, are given. We have merely to form the product of the two hypercomplex numbers; that is to say (*cf.* 4.4), we multiply two corresponding representations together as in ordinary algebra, substitute (5.2) or (5.3), and simplify. The validity of the process follows from the fact that it is simply a translation into symbols of the more self-explanatory procedure of chess-board diagrams: in other words, it follows from the fact that genetical multiplication obeys the distributive law.

## § 6. "SHORTCIRCUITED" MULTIPLICATION.

By a fundamental property of duplicate algebras (Etherington, 1941, -*Paper VII* Theorem I), multiplication in  $\mathbf{Z}$  can be "shortcircuited" by working in  $\mathbf{G}$ : that is to say, to find PQ when P and Q are given zygotically, we first apply (5.2) to obtain the gametic representations, and then multiply *without* applying (5.2). Similarly, to evaluate in  $\mathbf{Z}$  a complicated non-associative product involving any number of factors, all the operations can be performed in the simpler algebra  $\mathbf{G}$ , only the final product being left in quadratic form and interpreted as a hypercomplex number in  $\mathbf{Z}$ .

This corresponds to a well-known fact in genetics (*cf.* Jennings, 1917, pp. 101-102): in order to obtain the zygotic frequencies of an  $n$ th generation, provided that no selection acts on the zygotes, and in the absence of inbreeding, it is sufficient to trace only the gametic frequencies through the  $n-1$  intervening generations.

To consider, for example, random mating of a population  $P$  *inter se*, suppose

$$P = \delta D + \rho R, \quad (6.1)$$

where (5.9) holds if the zygotic representation is given in the first place. Then the next generation is

$$\left. \begin{aligned} F_1 = P^2 &= \delta^2 DD + 2\delta\rho DR + \rho^2 RR \\ &= \alpha_1 A + \beta_1 B + \gamma_1 C, \end{aligned} \right\} \quad (6.2)$$

where

$$\alpha_1 = \delta^2, \quad \beta_1 = 2\delta\rho, \quad \gamma_1 = \rho^2. \quad (6.3)$$

This is evidently simpler than evaluating  $P^2$  in  $\mathbf{Z}$  directly. The conclusion

$$\beta_1^2 = 4\alpha_1\gamma_1 \quad (6.4)$$

is the well-known Pearson-Hardy law.

Evaluating (6.2) in  $\mathbf{G}$  by use of (5.2),

$$\begin{aligned} P^2 &= \delta(\delta + \rho)D + \rho(\delta + \rho)R \\ &= \delta D + \rho R \end{aligned}$$

if (6.1) is normalised. Thus in  $\mathbf{G}$  any normalised hypercomplex number satisfies

$$P^2 = P; \quad (6.5)$$

hence all powers of  $P$  are equal, showing that the gene frequencies are undisturbed by random mating, or by random mating followed by any system of intermating of the generations. The zygotic distribution, however, in such cases, comes into equilibrium after one generation of random mating, since in  $\mathbf{Z}$  we find

$$P^3 = P^2, \quad P^{2.2} = P^2, \quad (6.6, 7)$$

and all higher powers of  $P$  are equal to  $P^2$ . These equations follow immediately from (6.5) if  $P^3$ ,  $P^{2.2}$  are found by short-circuited multiplication.

## § 7. THE MENDELIAN COPULAR ALGEBRA.

The procedure of duplication (5.10), by which  $\mathbf{Z}$  was derived from  $\mathbf{G}$ , can be applied to an algebra repeatedly. Let us form  $\mathbf{K}$ , the duplicate of  $\mathbf{Z}$ , and then consider its genetical significance. By analogy with (5.1) we begin by taking

$$AA, BB, CC, BC, CA, AB \quad (7.1)$$

as the units of a new algebra. There is no need to introduce fresh symbols. The multiplication table will consist of 21 equations derived by manipulation of the equations (5.3), for example:

$$(AA)^2 = AA, \quad (BB)^2 = \frac{1}{16}AA + \frac{1}{4}BB + \frac{1}{16}CC + \frac{1}{4}BC + \frac{1}{8}CA + \frac{1}{4}AB, \quad \text{etc.} \quad (7.2)$$

The interpretation is as follows: the coupled symbols (7.1) stand for the types of family into which the population can be sorted, classified according to the parental genotypes: or, we may say, they are the types of *couple* mated in the preceding generation. Hence (7.2) means that if a population of offspring of matings  $A \times A$  is mated at random with itself or with a similar population, all the matings are of this type  $A \times A$ ; but if the parental matings were all  $B \times B$ , then the six couple types occur in numerical proportions  $\frac{1}{16} : \frac{1}{4} : \dots$ ; and so on.

A population for which the relative frequencies of the couple types are

$$\lambda : \mu : \nu : \theta : \phi : \psi . \quad (7.3)$$

is represented by a hypercomplex number

(Copular representation)  $P = \lambda AA + \mu BB + \nu CC + \theta BC + \phi CA + \psi AB,$  (7.4)

wherein

(Normalising condition)  $\lambda + \mu + \nu + \theta + \phi + \psi = 1.$  (7.5)

From this we can pass by (5.3) and (5.1, 2) to the zygotic and gametic representations.

As in **G** and **Z**, the product of two hypercomplex numbers in **K** denoting populations gives in the same representation their offspring by random mating. This statement assumes that the couple types are not selected, *i.e.* they are of equal average surviving fertility; just as in **Z** and **G** we supposed no selection on zygotes or gametes. As before, multiplication in **K** can be short-circuited by working in **Z** or **G**.

Corresponding to the Pearson-Hardy law in the zygotic algebra, we have the following facts: a population, as a distribution of copular types, comes into equilibrium after *two* generations of amphimixis; after *one* generation, the equations

$$\theta^2 = 4\mu\nu, \quad \phi^2 = 4\nu\lambda, \quad \psi^2 = 4\lambda\mu \quad (7.6)$$

are satisfied; after two generations the further equation

$$\mu^2 = 16\nu\lambda \quad (7.7)$$

is satisfied; these four are the necessary and sufficient conditions for equilibrium in amphimixis, and imply also other relations such as

$$4\phi^2 = \theta\psi = \mu^2. \quad (7.8)$$

These results are obtained very simply by using short-circuited multiplication, observing that  $P^2$  is necessarily of the form  $(\alpha A + \beta B + \gamma C)^2$ , and the next generation  $P^{2.2}$  of the form  $((\delta D + \rho R)^2)$ .



## § 8. SYSTEMS OF MATING.

Four systems of mating will be considered. The object in each case is to obtain the distribution of types in a filial generation from the distribution in the preceding generation; also, when it can be done simply, to find the distribution in the  $n$ th filial generation, and the equilibrium distribution which this approaches as  $n$  increases. For other treatment of these and similar problems, cf. Jennings (1916, 1917), Wentworth and Remick (1916), Robbins (1917, 1918), Hogben (1931, Chap. 6; 1933), Geppert and Koller (1938, § 20).

(a) *Self-fertilisation, or Assortative Mating in Absence of Dominance.*

Starting from the zygotic distribution

$$P = aA + \beta B + \gamma C \quad . \quad . \quad . \quad (8a.1)$$

(where  $A = DD$ ,  $B = DR$ ,  $C = RR$ ), if mating proceeds in successive generations by self-fertilisation, or by each individual mating with another of the same type, the first filial generation  $F_1$  will consist of the offspring of  $A \times A$ ,  $B \times B$ ,  $C \times C$ , occurring in proportions  $a : \beta : \gamma$ ; so that

$$F_1 = aA^2 + \beta B^2 + \gamma C^2. \quad . \quad . \quad . \quad (8a.2)$$

$$= aA + \beta(\frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C) + \gamma C. \quad . \quad . \quad . \quad (8a.3)$$

$$= (a + \frac{1}{4}\beta)A + \frac{1}{2}\beta B + (\frac{1}{4}\beta + \gamma)C. \quad . \quad . \quad . \quad (8a.4)$$

It will be seen that the frequency of heterozygotes is halved; so if the  $n$ th filial generation is denoted

$$F_n = \alpha_n A + \beta_n B + \gamma_n C, \quad . \quad . \quad . \quad (8a.5)$$

we shall have

$$\beta_1 = \frac{1}{2}\beta, \quad \beta_2 = \frac{1}{4}\beta, \quad \beta_3 = \frac{1}{8}\beta, \quad . \quad . \quad ., \quad \beta_n = \frac{1}{2^n}\beta. \quad . \quad . \quad (8a.6)$$

Also

$$\alpha_1 = a + \frac{1}{4}\beta, \quad \gamma_1 = \frac{1}{4}\beta + \gamma. \quad . \quad . \quad . \quad (8a.7)$$

Let us find the quantities  $u_1, u_2, u_3, \dots$  by which the hypercomplex number representing the population increases in the successive generations. We have from (8a.1) and (8a.4):

$$u_1 = F_1 - P = \frac{1}{2}\beta(\frac{1}{2}A - B + \frac{1}{2}C); \quad . \quad . \quad . \quad (8a.8)$$

and similarly we shall have

$$u_2 = \frac{1}{2}\beta_1(\frac{1}{2}A - B + \frac{1}{2}C) = \frac{1}{4}\beta(\frac{1}{2}A - B + \frac{1}{2}C),$$

$$u_3 = \frac{1}{8}\beta(\frac{1}{2}A - B + \frac{1}{2}C), \quad . \quad . \quad ., \quad u_n = \frac{1}{2^n}\beta(\frac{1}{2}A - B + \frac{1}{2}C). \quad . \quad (8a.9)$$

The total increase in  $n$  generations is therefore

$$u_1 + u_2 + u_3 + \dots + u_n = \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} \right) \beta \left( \frac{1}{2}A - B + \frac{1}{2}C \right). \quad (8a.10)$$

The sum of the geometrical progression in brackets is  $1 - (\frac{1}{2})^n$ . Hence the  $n$ th filial generation is

$$F_n = aA + \beta B + \gamma C + \left( 1 - \frac{1}{2^n} \right) \beta \left( \frac{1}{2}A - B + \frac{1}{2}C \right) \quad (8a.11)$$

$$= \left( a + \frac{1}{2}\beta - \frac{1}{2^{n+1}}\beta \right) A + \frac{1}{2^n}\beta B + \left( \frac{1}{2}\beta + \gamma - \frac{1}{2^{n+1}}\beta \right) C. \quad (8a.12)$$

As the number of generations increases, this quickly approaches the limiting stable distribution

$$(a + \frac{1}{2}\beta)A + (\gamma + \frac{1}{2}\beta)C. \quad (8a.13)$$

(b) *Assortative Mating (Dominants  $\times$  Dominants, Recessives  $\times$  Recessives).*

The initial zygotic distribution

$$P = aA + \beta B + \gamma C \quad (8b.1)$$

may be written phenotypically

$$P = (a + \beta)\mathfrak{D} + \gamma C, \quad (8b.2)$$

Here

$$\mathfrak{D} = \frac{a}{a + \beta}A + \frac{\beta}{a + \beta}B = \frac{aA + \beta B}{a + \beta}, \quad (8b.3)$$

representing the genotype distribution of the dominants in  $P$ .

With the system of mating under consideration, the first filial generation is

$$F_1 = (a + \beta)\mathfrak{D}^2 + \gamma C^2 \quad (8b.4)$$

$$= \frac{(aA + \beta B)^2}{a + \beta} + \gamma C^2. \quad (8b.5)$$

Therefore

$$\begin{aligned} (a + \beta)F_1 &= (a^2A^2 + 2a\beta AB + \beta^2B^2) + (a + \beta)\gamma C^2 \\ &= a^2A + 2a\beta \left( \frac{1}{2}A + \frac{1}{2}B \right) + \beta^2 \left( \frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C \right) + (a + \beta)\gamma C \\ &= (a^2 + a\beta + \frac{1}{4}\beta^2)A + (a\beta + \frac{1}{2}\beta^2)B + (\frac{1}{4}\beta^2 + a\gamma + \beta\gamma)C. \end{aligned} \quad (8b.6)$$

It will be found that  $F_1 - P$  is a multiple of  $\frac{1}{2}A - B + \frac{1}{2}C$ , and hence that  $F_n$  can be found by summation of a series, just as in Case (a). The series in this case is not a geometrical progression, but it is of a type whose sum is easily obtained. Following the procedure of Case (a), it will be found that the total increase in  $n$  generations can be expressed as

$$\begin{aligned}
& \left\{ \frac{\frac{1}{2}\beta}{(a+\frac{1}{2}\beta)(a+\beta)} + \frac{\frac{1}{2}\beta}{(a+\beta)(a+\frac{3}{2}\beta)} + \dots + \frac{\frac{1}{2}\beta}{(a+\frac{1}{2}n\beta)(a+\frac{1}{2}(n+1)\beta)} \right\} \beta(a+\frac{1}{2}\beta)(\frac{1}{2}A-B+\frac{1}{2}C) \\
&= \left\{ \left( \frac{1}{a+\frac{1}{2}\beta} - \frac{1}{a+\beta} \right) + \left( \frac{1}{a+\beta} - \frac{1}{a+\frac{3}{2}\beta} \right) + \dots + \left( \frac{1}{a+\frac{1}{2}n\beta} - \frac{1}{a+\frac{1}{2}(n+1)\beta} \right) \right\} \beta(a+\frac{1}{2}\beta)(\frac{1}{2}A-B+\frac{1}{2}C) \\
&= \left\{ \frac{1}{a+\frac{1}{2}\beta} - \frac{1}{a+\frac{1}{2}(n+1)\beta} \right\} \beta(a+\frac{1}{2}\beta)(\frac{1}{2}A-B+\frac{1}{2}C) \\
&= \left\{ \beta - \frac{\beta(a+\frac{1}{2}\beta)}{a+\frac{1}{2}(n+1)\beta} \right\} (\frac{1}{2}A-B+\frac{1}{2}C). \quad (8b.7)
\end{aligned}$$

The  $n$ th filial generation is obtained by adding (8b.7) to (8b.1). We obtain

$$\begin{aligned}
F_n &= aA + \beta B + \gamma C + \beta(\frac{1}{2}A-B+\frac{1}{2}C) - \frac{\beta(a+\frac{1}{2}\beta)}{a+\frac{1}{2}(n+1)\beta}(\frac{1}{2}A-B+\frac{1}{2}C) \\
&= (a+\frac{1}{2}\beta)A + (\frac{1}{2}\beta+\gamma)C - \frac{\beta(a+\frac{1}{2}\beta)}{a+\frac{1}{2}(n+1)\beta}(\frac{1}{2}A-B+\frac{1}{2}C). \quad (8b.8)
\end{aligned}$$

As  $n$  increases, the fraction with  $a+\frac{1}{2}(n+1)\beta$  in the denominator approaches zero. Hence  $F_n$  approaches a stable distribution, namely,  $(a+\frac{1}{2}\beta)A + (\frac{1}{2}\beta+\gamma)C$ , the same as in Case (a). (Cf. 8a.13.)

### (c) Fraternal Mating.

In this and the following case it is necessary to use the copular representation (7.4), from which of course the zygotic representation can be deduced. The determination of  $F_n$  is much more difficult than in Cases (a) and (b). It is best obtained with the aid of matrix algebra; and as this is beyond the scope of this paper, I content myself with showing only in each case how the copular representation of any generation is deduced from the preceding.

Suppose that initially

$$P = \lambda AA + \mu BB + \nu CC + \theta BC + \phi CA + \psi AB, \quad (8c.1)$$

and that brothers and sisters are mated at random. Then the filial generation is

$$F_1 = \lambda(AA)^2 + \mu(BB)^2 + \nu(CC)^2 + \theta(BC)^2 + \phi(CA)^2 + \psi(AB)^2. \quad (8c.2)$$

Using short-circuited multiplication (*i.e.* (5.3) instead of (7.2)),

$$\begin{aligned}
F_1 &= \lambda(A)^2 + \mu(\frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C)^2 + \nu(C)^2 + \theta(\frac{1}{2}B + \frac{1}{2}C)^2 + \phi(B)^2 + \psi(\frac{1}{2}A + \frac{1}{2}B)^2 \\
&= \lambda AA + \mu(\frac{1}{16}AA + \frac{1}{4}BB + \frac{1}{8}CC + \frac{1}{4}BC + \frac{1}{8}CA + \frac{1}{4}AB) + \nu CC \\
&\quad + \theta(\frac{1}{4}BB + \frac{1}{2}BC + \frac{1}{4}CC) + \phi BB + \psi(\frac{1}{4}AA + \frac{1}{2}AB + \frac{1}{4}BB) \\
&= (\lambda + \frac{1}{16}\mu + \frac{1}{4}\psi)AA + (\frac{1}{4}\mu + \frac{1}{4}\theta + \phi + \frac{1}{4}\psi)BB + (\frac{1}{8}\mu + \nu + \frac{1}{4}\theta)CC \\
&\quad + (\frac{1}{4}\mu + \frac{1}{2}\theta)BC + \frac{1}{8}\mu CA + (\frac{1}{4}\mu + \frac{1}{2}\psi)AB. \quad (8c.3)
\end{aligned}$$

(d) *Filial Mating.*

Starting from an arbitrary copular distribution as in (c), suppose that each individual (or each individual of one sex) is mated with the parent of opposite sex. Then

$$\begin{aligned} F_1 &= \lambda AA.A + \mu BB.B + \nu CC.C + \theta BC.(\tfrac{1}{2}B + \tfrac{1}{2}C) + \phi CA.(\tfrac{1}{2}C + \tfrac{1}{2}A) \\ &\quad + \psi AB.(\tfrac{1}{2}A + \tfrac{1}{2}B) \quad (8d.1) \\ &= \lambda A.A + \mu(\tfrac{1}{4}A + \tfrac{1}{2}B + \tfrac{1}{4}C)B + \nu C.C + \theta(\tfrac{1}{2}B + \tfrac{1}{2}C)^2 + \phi B(\tfrac{1}{2}C + \tfrac{1}{2}A) + \psi(\tfrac{1}{2}A + \tfrac{1}{2}B)^2 \\ &= \lambda AA + \mu(\tfrac{1}{4}AB + \tfrac{1}{2}BB + \tfrac{1}{4}BC) + \nu CC + \theta(\tfrac{1}{4}BB + \tfrac{1}{2}BC + \tfrac{1}{4}CC) \\ &\quad + \phi(\tfrac{1}{2}BC + \tfrac{1}{2}AB) + \psi(\tfrac{1}{4}AA + \tfrac{1}{2}AB + \tfrac{1}{4}BB) \\ &= (\lambda + \tfrac{1}{4}\psi)AA + (\tfrac{1}{2}\mu + \tfrac{1}{4}\theta + \tfrac{1}{4}\psi)BB + (\nu + \tfrac{1}{4}\theta)CC + (\tfrac{1}{4}\mu + \tfrac{1}{2}\theta + \tfrac{1}{2}\phi)BC \\ &\quad + \phi.CA + (\tfrac{1}{4}\mu + \tfrac{1}{2}\phi + \tfrac{1}{2}\psi)AB. \quad (8d.2) \end{aligned}$$

§ 9. COMPACT MULTIPLICATION TABLES.

If P and Q are any two normalised hypercomplex numbers in **G** (say  $\delta D + \rho R$ ,  $\delta'D + \rho'R$ , where  $\delta + \rho = \delta' + \rho' = 1$ ), then

$$PQ = \tfrac{1}{2}P + \tfrac{1}{2}Q. \quad (9.1)$$

This may be shown directly by multiplying and applying (5.2); or more briefly by observing that  $\tfrac{1}{2}P + \tfrac{1}{2}Q$  is also normalised, so that by (6.5)

$$\begin{aligned} P^2 &= P, \quad Q^2 = Q, \\ \tfrac{1}{2}P + \tfrac{1}{2}Q &= (\tfrac{1}{2}P + \tfrac{1}{2}Q)^2 = \tfrac{1}{4}P^2 + \tfrac{1}{2}PQ + \tfrac{1}{4}Q^2 = \tfrac{1}{4}P + \tfrac{1}{2}PQ + \tfrac{1}{4}Q, \end{aligned}$$

from which (9.1) follows.

The result (9.1) may be regarded as a compact form of the gametic multiplication table, since it includes the three equations (5.2) as special cases. (It must be noted that (9.1) only applies if P, Q are normalised. The more general result is:  $PQ = \tfrac{1}{2}(\delta' + \rho')P + \tfrac{1}{2}(\delta + \rho)Q$ .)

It will be convenient to use the letters *a*, *b*, *c*, . . . to denote each either D or R, and then the compact multiplication table may be written

$$ab = \tfrac{1}{2}a + \tfrac{1}{2}b. \quad (9.2)$$

Applying to this the process of duplication, we obtain

$$\begin{aligned} ab.cd &= (\tfrac{1}{2}a + \tfrac{1}{2}b)(\tfrac{1}{2}c + \tfrac{1}{2}d), \\ i.e. \quad ab.cd &= \tfrac{1}{4}ac + \tfrac{1}{4}ad + \tfrac{1}{4}bc + \tfrac{1}{4}bd, \quad (9.3) \end{aligned}$$

which gives a compact form of the zygotic multiplication table: for it includes all the six equations (5.3) as special cases. *E.g.* on putting  $a=c=D$ ,  $b=d=R$ , we get from (9.3) the formula for  $B^2$ .

Similarly, the compact copular multiplication table is

$$ab.cd : ef.gh = \frac{1}{16}(a+b)(c+d).(e+f)(g+h),$$

i.e.

$$ab.cd : ef.gh = \frac{1}{16}ac.eg + \frac{1}{16}ac.eh + \dots \quad (16 \text{ terms}), \quad (9.4)$$

which includes the 21 equations (7.2).

#### § 10. OFFSPRING OF CONSANGUINEOUS MARRIAGES.

Formulae for the probability of RR offspring of various kinds of consanguineous marriages were given by Dahlberg (1929), and his verbal arguments may be translated into non-associative algebra. As an example, consider the distribution of genotypes DD, DR, RR in the offspring of a marriage between first cousins. This distribution will be found from a non-associative product of the form

$$ab.cd : ef : ab.cd : gh. \quad (10.1)$$

Here each of the letters stands for either D or R;  $ab$  and  $cd$  denote the genetic constitutions of the common grandparents of the cousins; the two sibs, parents of cousins, are both represented by  $ab.cd$ ; and the cousins themselves by  $ab.cd : ef$  and  $ab.cd : gh$  respectively.

Simplifying (10.1) by repeated use of (9.1), we have

$$\begin{aligned} ab.cd : ef &= \left\{ \frac{1}{2} \left( \frac{a+b}{2} + \frac{c+d}{2} \right) \right\} \frac{e+f}{2} = \left( \frac{a+b+c+d}{4} \right) \left( \frac{e+f}{2} \right) \\ &= \frac{1}{2} \left( \frac{a+b+c+d}{4} + \frac{e+f}{2} \right) = \frac{1}{8}(a+b+c+d+2e+2f). \end{aligned}$$

Similarly,

$$ab.cd : gh = \frac{1}{8}(a+b+c+d+2g+2h).$$

Therefore

$$ab.cd : ef : ab.cd : gh = \frac{1}{8}(a^2+b^2+c^2+d^2+60 \text{ product terms}). \quad (10.2)$$

We must now take into account whatever information is given about the genetic constitution of the four grandparents. We might, for example, be given the genotype of one of them. Assuming, however, that they are merely random members of a stable population,

$$P = \delta^2 DD + 2\delta\rho DR + \rho^2 RR = \delta D + \rho R, \quad (\delta + \rho = 1)$$

then the probability of  $a$  being D or R is  $\delta$  or  $\rho$ , and so for each of the ancestral gametes. Hence (10.2) yields for the offspring of first cousins the probability distribution

$$\begin{aligned} &\frac{1}{8} \{ (4\delta + 60\delta^2) DD + 60\delta\rho DR + (4\rho + 60\rho^2) RR \} \\ &= \left( \frac{1}{16}\delta + \frac{1}{16}\delta^2 \right) DD + \frac{1}{8}\delta\rho DR + \left( \frac{1}{16}\rho + \frac{1}{16}\rho^2 \right) RR, \quad (10.3) \end{aligned}$$

agreeing with Dahlberg's result.

$$\begin{array}{c} (ab)(cd) \\ \downarrow \quad \downarrow \\ (ef) \times (ab.cd) \quad (ab.cd) \times (gh) \\ \downarrow \quad \downarrow \\ (ab.cd : ef) \dots \times \dots (ab.cd : gh) \\ \downarrow \quad \downarrow \\ (ab.cd : ef : ab.cd : gh) \end{array}$$



§ 11. FURTHER GENETIC ALGEBRAS.

Consider inheritance depending on two pairs of autosomal allelomorphs, say D, R and D', R'. The corresponding gametic algebras **G**, **G'** have multiplication tables:

$$\left. \begin{array}{lll} D^2 = D, & DR = \frac{1}{2}D + \frac{1}{2}R, & R^2 = R; \\ D'^2 = D', & D'R' = \frac{1}{2}D' + \frac{1}{2}R', & R'^2 = R'. \end{array} \right\} \quad (11.1)$$

Taking both pairs into account, there are four gametic types:

$$DD', \quad DR', \quad RD', \quad RR', \quad (11.2)$$

whose multiplication table is constructed as follows:—

$$\left. \begin{array}{l} DD' \cdot DD' = D^2 \cdot D'^2 = DD', \\ DD' \cdot DR' = D^2 \cdot D'R' = D(\frac{1}{2}D' + \frac{1}{2}R') = \frac{1}{2}DD' + \frac{1}{2}DR', \\ DR' \cdot RD' = DR \cdot D'R' = (\frac{1}{2}D + \frac{1}{2}R)(\frac{1}{2}D' + \frac{1}{2}R') = \frac{1}{4}DD' + \frac{1}{4}DR' + \frac{1}{4}RD' + \frac{1}{4}RR', \end{array} \right\} \quad (11.3)$$

and so on. (10 equations.)

(It will be seen that although multiplication is non-associative we assume, *e.g.*,  $DD' \cdot DR' = D^2 \cdot D'R'$ . This is justified because the combination of dashed and undashed symbols is mere juxtaposition, not genetical multiplication.) This is precisely the process referred to in § 4 of forming the direct product of the two algebras **G**, **G'**, which is well known in the theory of linear algebras.

Alternatively, let us use  $a, b$  to denote each either D or R, and  $a', b'$  similarly for D' or R', so that, for example,  $aa'$  can denote any of the four gametic types. Then we can write the joint multiplication table in the compact form:

$$aa' \cdot bb' = ab \cdot a'b' = (\frac{1}{2}a + \frac{1}{2}b)(\frac{1}{2}a' + \frac{1}{2}b'), \quad (11.4)$$

*i.e.*

$$aa' \cdot bb' = \frac{1}{4}aa' + \frac{1}{4}ab' + \frac{1}{4}ba' + \frac{1}{4}bb'. \quad (11.5)$$

The zygotic algebra is obtained by duplicating the gametic, and is the direct product **ZZ'**. That is to say, it is immaterial whether the process of duplication is carried out before or after that of forming the direct product (Etherington, 1941, Theorem V). There is one point, in this connection, which requires elucidation. It has been pointed out that by pairing the four gametic types (11.2) we obtain the ten types of zygote, namely:

$$\begin{array}{llll} DD' \cdot DD', & DD' \cdot DR', & DD' \cdot RD', & DD' \cdot RR', \\ & DR' \cdot DR', & DR' \cdot RD', & DR' \cdot RR', \\ & RD' \cdot RD', & RD' \cdot RR', & RR' \cdot RR', \end{array} \quad (11.6)$$

which figure in (11.3). There are, however, only nine genotypes, namely:

$$\begin{array}{llll} DDD'D', & DDD'R', & DDR'R', & DRD'D', \\ & DRD'R', & DRR'R', & RRD'D', \\ & & RRD'R', & RRR'R', \end{array} \quad (11.7)$$

(Paper V 11)

I am indebted to Dr J. Ffoulkes Edwards for a lengthy correspondence in which this paper germinated; and to Dr Charlotte Auerbach, of the Institute of Animal Genetics, University of Edinburgh, for much constructive criticism.

#### REFERENCES TO LITERATURE.

- DAHLBERG, G., 1929. "Inbreeding in man," *Genetics*, vol. xiv, pp. 421-454.  
 ETHERINGTON, I. M. H., 1939 *a*. "On non-associative combinations," *Proc. Roy. Soc. Edin.*, vol. lix, pp. 153-162.  
 —, 1939 *b*. "Genetic algebras," *ibid.*, pp. 242-258. (Referred to as "G.A.")  
 —, 1941. "Duplication of linear algebras," *Proc. Edin. Math. Soc.* (*In press.*)  
 GEPPERT, H., and KOLLER, S., 1938. *Erbmathematik*, Leipzig.  
 HOGGEN, L., 1931. *Genetic Principles in Medicine and Social Science*, London.  
 —, 1933. "A matrix notation for mendelian populations," *Proc. Roy. Soc. Edin.*, vol. liii, pp. 7-25.  
 JENNINGS, H. S., 1916, 1917. "The numerical results of diverse systems of breeding," *Genetics*, vol. i, pp. 53-89; vol. ii, pp. 97-154.  
 ROBBINS, R. B., 1917, 1918. "Some applications of mathematics to breeding problems," *Genetics*, vol. ii, pp. 489-504; vol. iii, pp. 73-92, 375-389.  
 WENTWORTH, E. N., and REMICK, B. L., 1916. "Some breeding properties of the generalised mendelian population," *Genetics*, vol. i, pp. 608-616.

(Issued separately March 18, 1941.)

PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH

SESSION 1938-1939

VOL. LIX—PART III—(No. 23)

XXIII.—Genetic Algebras. By I. M. H. Etherington, B.A.(Oxon),  
Ph.D.(Edin.), Mathematical Institute, University of Edinburgh.

(MS. received May 8, 1939. Revised MS. received September 5, 1939.  
Read July 3, 1939.)

CONTENTS.

|                                 | PAGE |                                 | PAGE |
|---------------------------------|------|---------------------------------|------|
| § 1. INTRODUCTION . . . . .     | 242  | §§ 10-15. EXAMPLES OF SYM-      |      |
| § 2. NOTATION . . . . .         | 243  | METRICAL GENETIC ALGE-          |      |
| § 3. BARIC ALGEBRAS . . . . .   | 244  | BRAS . . . . .                  | 252  |
| § 4. TRAIN ALGEBRAS . . . . .   | 245  | § 10. SIMPLE MENDELIAN INHERIT- |      |
| § 5. DUPLICATION . . . . .      | 248  | ANCE . . . . .                  | 253  |
| § 6. GAMETIC ALGEBRAS . . . . . | 249  | § 11. MULTIPLE ALLELOMORPHS .   | 254  |
| § 7. ZYGOTIC ALGEBRAS . . . . . | 250  | § 12. LINKED ALLELOMORPHS .     | 254  |
| § 8. FURTHER DUPLICATE GEN-     |      | § 13. INDEPENDENT ALLELOMORPHS  | 254  |
| ETIC ALGEBRAS . . . . .         | 251  | § 14. LINKAGE GROUP . . . . .   | 255  |
| § 9. COMBINATION OF GENETIC     |      | § 15. POLYPLOIDY . . . . .      | 257  |
| ALGEBRAS . . . . .              | 251  | SUMMARY . . . . .               | 258  |
|                                 |      | REFERENCES TO LITERATURE        | 258  |

§ 1. INTRODUCTION.

TWO classes of linear algebras, generally non-associative, are defined in § 3 (*baric algebras*) and § 4 (*train algebras*), and the process of *duplication* of a linear algebra in § 5. These concepts, which will be discussed more fully elsewhere, arise naturally in the symbolism of genetics, as shown in §§ 6-15. Many of their properties express facts well known in genetics; and the processes of calculation which are fundamental in many problems of population genetics can be expressed as manipulations in the genetic algebras. In cases where inheritance is of a simple type (*e.g.* §§ 10-13, 15) this constitutes a new point of view, but perhaps amounts to little more than a change of notation as compared with existing methods. § 14, however, indicates the possibility of generalisations which would seem to be impossible by ordinary methods.

The occurrence of the genetic algebras may be described in general terms as follows. The mechanism of chromosome inheritance, in so far as it determines the probability distributions of genetic types in families and filial generations, and expresses itself through their frequency distributions, may be represented conveniently by algebraic symbols. Such a symbolism is described, for instance, by Jennings (1935, chap. ix);

many applications are given by Geppert and Koller (1938). It is shown in the present paper that the symbolism is equivalent to the use of a system of related linear algebras, in which multiplication (equivalent to the procedure of "chessboard diagrams") is commutative ( $PQ = QP$ ) but non-associative ( $PQ \cdot R \neq P \cdot QR$ ). A population (*i.e.* a distribution of genetic types) is represented by a normalised hypercomplex number in one or other algebra, according to the point of view from which it is specified. If  $P$ ,  $Q$  are populations, the filial generation  $P \times Q$  (*i.e.* the statistical population of offspring resulting from the random mating of individuals of  $P$  with individuals of  $Q$ ) is obtained by multiplying two corresponding representations of  $P$  and  $Q$ ; and from this requirement of the symbolism it will be obvious why multiplication must be non-associative. It must be understood that a population may mean a single individual, or rather the information which we may have concerning him in the form of a probability distribution.

Inheritance will be called *symmetrical* if the sex of a parent does not affect the distribution of gametic types produced. Paying attention only to the inheritance of gene differences (not of phenotypes), every regular mode of symmetrical inheritance in theoretical genetics has its fundamental *gametic algebra*, from which other algebras (*zygotic*, etc.) are deduced by duplication. From the nature of the symbolism these are of necessity baric algebras; but it appears on closer examination that they belong in all cases to the narrower category of train algebras.

(The fundamental algebras can be modified to take account of various kinds of selection. They are then no longer train algebras, although the baric property and the relation of duplication sometimes persist.)

Symmetry of inheritance may be disturbed by unequal crossing over in male and female, by sex linkage, or by gametic selection. These cases are not discussed at all in the present paper; but it may be stated briefly\* that in the absence of selection the corresponding genetic algebras (of order  $n$ , say) possess train subalgebras (of order  $n-1$ ).

The occurrence of a non-associative linear algebra in the simplest case of Mendelian inheritance was pointed out by Glivenko (1936).

## § 2. NOTATION.

By *principal powers* in a non-associative algebra, I mean powers in which the factors are absorbed one at a time always on the right or always on the left (see (3.6)). Otherwise, for the notation and nomenclature for non-associative products and powers, see my paper "On Non-Associative Combinations" (1939). The word *pedigree* which occurs there can now be interpreted almost in its ordinary biological sense.

---

\* Cf. the postscript to paper IX.

Elements of a linear algebra (*i.e.* hypercomplex numbers) will be called *elements* and denoted by Latin letters, generally small ( $a, b, \dots$ ); but normalised elements, *i.e.* elements of unit weight (§ 3), will be denoted by Latin capitals ( $A, B, \dots$ ). The letters  $m, n, r$ , however, denote positive integers.

Elements of the field  $F$  over which a linear algebra is defined will be called *numbers* and denoted by small Greek letters ( $\alpha, \beta, \dots$ ). Thus, an *element* is determined by its coefficients, which are *numbers*. In the genetical applications,  $F$  may be taken as the field of real numbers. The enumerating indices (subscripts and superscripts) take positive integer values, either 1 to  $m$ , 1 to  $n$ , or 1 to  $r$ , according to the context.

Block capitals ( $A, B, \dots$ ) denote algebras.

The symbol  $\Sigma$  indicates summation with respect to repeated indices, *e.g.* with respect to  $\sigma$  in (3.3), with respect to  $\sigma$  and  $\tau$  in (5.3).

The symbol  $1^\mu$  stands for a set of 1's. Thus the formula (6.3) means the same as

$$\sum_{\sigma=1}^n \gamma_{\sigma}^{\mu\nu} = 1.$$

The advantage of this notation is that such formulæ retain their form under linear transformations of the basis of a genetic algebra,  $1^\mu$  being replaced by the vector  $\xi^\mu$  (*cf.* (6.12)).

### § 3. BARIC ALGEBRAS.

It is well known that a linear associative algebra possesses a matrix representation. Non-associative algebras in general do not, but may. The simplest such representation would be a scalar representation on the field  $F$  over which the algebra is defined. A linear algebra  $X$ , associative or not, which possesses a non-trivial representation of this kind, will be called *baric*.

The definition means that to any element  $x$  of  $X$  there corresponds a number  $\xi(x)$  of  $F$ , not identically zero, such that

$$\xi(x+y) = \xi(x) + \xi(y), \quad \xi(ax) = a\xi(x), \quad \xi(xy) = \xi(x)\xi(y). \quad (x, y \in X, a \in F) \quad (3.1)$$

$\xi(x)$  will be called the *weight* of  $x$ , or the *weight function* of  $X$ . If  $\xi(x) \neq 0$ ,  $x$  can be *normalised*—that is, replaced by the element

$$X = x/\xi(x) \quad (3.2)$$

of unit weight. Elements of zero weight will be called *nil elements*. The set  $U$  of all nil elements is evidently an invariant subalgebra of  $X$ ; *i.e.*  $XU \leq U$ : it will be called the *nil subalgebra*.



Let the multiplication table of a linear algebra  $\mathbf{X}$  be

$$a^\mu a^\nu = \sum \gamma_{\sigma}^{\mu\nu} a^\sigma, \quad (\mu, \nu, \sigma = 1, \dots, n) \quad (3.3)$$

and let the general element be denoted

$$x = \sum a_\mu a^\mu. \quad (3.4)$$

For  $\mathbf{X}$  to be a baric algebra, it is necessary and sufficient that the equations (3.3), regarded as ordinary simultaneous equations in  $\mathbf{F}$  for the unknowns  $a^\mu$ , should possess a non-null solution  $a^\mu = \xi^\mu$ . For this is obviously necessary, the  $\xi^\mu$  being the weights of the basic elements  $a^\mu$ . Conversely, if the condition is satisfied and we take

$$\xi(x) = \sum a_\mu \xi^\mu, \quad (3.5)$$

then (3.1) are at once deducible. The basic weights  $\xi^\mu$  form the *weight vector* of  $\mathbf{X}$ . In the genetical applications,  $\xi^\mu = 1^\mu$ .

Let the right rank equation (Dickson, 1914, § 19), or equation of lowest degree connecting the right principal powers,

$$x, x^2, x^3, \dots, x^m = x^{m-1}x, \dots, \quad (3.6)$$

be

$$f(x) \equiv x^r + \theta_1 x^{r-1} + \theta_2 x^{r-2} + \dots + \theta_{r-1} x = 0, \quad (3.7)$$

where each coefficient  $\theta_m$  is a homogeneous polynomial of degree  $m$  in the co-ordinates  $a_\mu$  of  $x$ . Then  $f(x)$ , being zero, is of zero weight. Hence the equation is satisfied when we substitute  $\xi(x)$  for  $x$ ; consequently  $x - \xi(x)$  must be a factor of  $f(x)$ . The same is true for the left rank equation. Thus

$$\xi(x) \text{ is a root of the right and left rank equations.} \quad (3.8)$$

The weight function of an algebra is not necessarily unique. In fact, a commutative associative linear algebra for which the determinant  $|\sum \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\sigma\tau}|$  does not vanish has  $n$  independent weight functions; and its rank equation is hence completely determined by (3.8) (Dickson, 1914, § 55, and the references given there).

#### § 4. TRAIN ALGEBRAS.

A baric algebra with the weight function  $\xi(x)$  and right rank equation (3.7) will be called a *right train algebra* if the coefficients  $\theta_m$ , in so far as they depend on the element  $x$ , depend only on  $\xi(x)$ . A *left train algebra* is defined similarly. For simplicity, suppose multiplication commutative, so that we may drop "left" and "right."

Since  $\theta_m$  is homogeneous of degree  $m$  in the co-ordinates of  $x$ , it must in a train algebra be a numerical multiple of  $\xi(x)^m$ . Hence (if the field  $\mathbf{F}$

be sufficiently extended, *e.g.*, to include complex numbers) the rank equation can be factorised:

$$f(x) \equiv x(x - \xi)(x - \lambda_1 \xi)(x - \lambda_2 \xi) \dots = 0. \quad (4.1)$$

(It is implied that when the left side is expanded, powers of  $x$  are interpreted as principal powers.) The numbers  $1, \lambda_1, \lambda_2 \dots$  are the *principal train roots* of the algebra.

For a normalised element (3.7) becomes

$$f(X) \equiv X^r + \theta_1 X^{r-1} + \theta_2 X^{r-2} + \dots + \theta_{r-1} X = 0, \quad (4.2)$$

where now the  $\theta$ 's are constant (*i.e.* independent of  $X$ ); and (4.1) becomes

$$f(X) \equiv X(X - 1)(X - \lambda_1)(X - \lambda_2) \dots = 0. \quad (4.3)$$

Since (4.2) can be multiplied by  $X$  any number of times, it can be regarded as a linear recurrence equation with constant coefficients connecting the principal powers of the general normalised element  $X$ . Solving the recurrence relation for  $X^m (m \geq r)$  in the usual way, we obtain  $1, \lambda_1, \lambda_2 \dots$  as the roots of the auxiliary equation; hence a formula for  $X^m$  can be written down in terms of  $X, X^2, \dots, X^{r-1}$ . Hence also for the general non-nil element  $x = \xi X$ , the value of  $x^m = \xi^m X^m$  is known; while for a nil element  $u, u^m = 0 (m \geq r)$ .

The properties of train algebras will be studied elsewhere, and the following theorem proved:— (Paper IX, §§ 3, 4)

If (1)  $X$  is a baric algebra; (2) its nil subalgebra  $U$  is nilpotent (Wedderburn, 1908 a, p. 111); (3) for  $m = 1, 2, 3, \dots$ , the subalgebra  $U^{(m)}$ , consisting of all products of altitude  $m$  (Etherington, 1939, p. 156) (Paper II) formed from nil elements is an invariant subalgebra of  $X$  (as it necessarily is of  $U$ ); then  $X$  is a train algebra.

~~For train algebras of rank  $r = 2$  or  $3$ , provided that the principal train roots do not include  $\frac{1}{2}$ , the conditions are necessary as well as sufficient; but I cannot say whether this converse holds more generally or not. I will call  $X$  a special train algebra if it satisfies the conditions (1), (2), (3). In such algebras it can be shown that there are many other sequences which have properties like those of the sequence of principal powers; *i.e.* sequences of elements derived from the general element, which satisfy linear recurrence equations whose coefficients, being functions of the weight only, become constants on normalisation. Such sequences will be called *trains*. For example, the sequence of plenary powers~~

$$x, x^2, x^{2^2}, x^{2^3}, \dots, \quad (4.4)$$

and the sequence of primary products

$$x, Yx, Y.Yx, Y:Y.Yx, \dots, \quad (4.5)$$

form trains in a special train algebra. [Cf. Paper VII, p. 145.]

It is convenient to denote the  $m$ th element of a train as  $x^{[m]}$ , and to regard it as a symbolic  $m$ th power of  $x$ . Let the normalised recurrence equation, or *train equation*, be

$$g[X] \equiv X^{[s]} + \phi_1 X^{[s-1]} + \phi_2 X^{[s-2]} + \dots + \phi_{s-1} X = 0, \quad (4.6)$$

where the  $\phi$ 's are numerical constants. It is implied that the equation may be symbolically "multiplied all through" by  $X$  any number of times. It may also be symbolically factorised:

$$g[X] \equiv X[X - \mu_1][X - \mu_2] \dots = 0. \quad (4.7)$$

The square brackets indicate that after expansion powers of  $X$  are to be interpreted as symbolic powers. The expansion being performed as in ordinary algebra, multiplication of the symbolic factors is commutative and associative. Extra factors may be introduced without destroying the validity of the train equation; but assuming that all superfluous factors have been removed,  $s$  is the *rank* of the train, and the numbers  $1, \mu_1, \mu_2, \dots$  are the *train roots*, by means of which a formula for  $X^{[m]}$  ( $m \geq s$ ) can be written down.

In the applications to genetics, it will be found that all the fundamental symmetrical genetic algebras are special train algebras. Various trains have genetical significance; the  $X^{[m]}$  represent successive discrete generations of an evolving population or breeding experiment, and the train equation is the recurrence equation which connects them. (i.e. genetic)

Thus, for example, plenary powers (4.4) refer to a population with random mating; principal powers (3.6) to a mating system in which each generation is mated back to one original ancestor or ancestral population; and the primary products (4.5) to the descendants of a single individual or subpopulation  $X$  mating at random within a population  $Y$ . Other mating systems are described by other sequences, and in various well-known cases these have the train property—that is, the determination of the  $m$ th generation depends ultimately on a linear recurrence equation with constant coefficients. It usually happens that the train roots are real, distinct, and not exceeding unity. Hence it may be shown that  $X^{[m]}$  tends to equilibrium with increasing  $m$ ; the rate of approach to equilibrium is ultimately that of a geometrical progression with common ratio equal to the largest train root excluding unity; but it may be some generations (depending on the number of train roots) before this rate of approach is manifest.

Train roots may be described as the eigen-values of the operation of symbolic multiplication by  $X$ , or in genetic language, the operation of passing from one generation to the next.

Train algebras of (principal) rank 3, which occur in several contexts

in genetics, have certain special properties. For example, if the train equation for principal powers is  $X(X-1)(X-\lambda)=0$ , then the train equation for plenary powers is  $X[X-1][X-2\lambda]=0$ ; and vice versa. Examples may be seen below in (10.12), (12.4, 5), (15.3), where respectively  $\lambda=0, \frac{1}{2}(1-\omega), \frac{1}{6}$ .

(Paper VIII)  
Theorem XII

### § 5. DUPLICATION.

Let

$$a^\mu a^\nu = \sum \gamma_{\sigma}^{\mu\nu} a^\sigma \quad (5.1)$$

be the multiplication table of a linear algebra  $\mathbf{X}$  with basis  $a^\mu$  ( $\mu=1, \dots, n$ ). Then

$$a^\mu a^\nu \cdot a^\theta a^\phi = \sum \gamma_{\sigma}^{\mu\nu} a^\sigma \cdot \sum \gamma_{\tau}^{\theta\phi} a^\tau.$$

Writing

$$a^\mu a^\nu = a^{\mu\nu}, \quad (5.2)$$

this becomes

$$a^{\mu\nu} a^{\theta\phi} = \sum \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\theta\phi} a^{\sigma\tau}, \quad (5.3)$$

which may be regarded as the multiplication table of another linear algebra, isomorphic with the totality of quadratic forms in the original algebra. It will be called the *duplicate* of  $\mathbf{X}$ , and denoted  $\mathbf{X}'$ . It is commutative and of order  $\frac{1}{2}n(n+1)$  if  $\mathbf{X}$  is commutative; non-commutative and of order  $n^2$  if  $\mathbf{X}$  is non-commutative. It is generally non-associative, even if  $\mathbf{X}$  is associative. It is not to be confused with what may be called the *direct square* of  $\mathbf{X}$ , or direct product of two algebras isomorphic with  $\mathbf{X}$ : this would be an algebra of order  $n^2$ , having the multiplication table

$$a^{\mu\nu} a^{\theta\phi} = \sum \gamma_{\sigma}^{\mu\theta} \gamma_{\tau}^{\nu\phi} a^{\sigma\tau}, \quad (5.4)$$

differing from (5.3) in the arrangement of indices.

Some theorems on duplication will be proved elsewhere. It will be shown that the duplicates (i) of a linear transform of an algebra, (ii) of the direct product of two algebras, (iii) of a baric algebra with weight vector  $\xi^\mu$ , (iv) of a train algebra with principal train roots  $1, \lambda, \mu, \dots$ , are respectively (i) a linear transform of the duplicate algebra, (ii) the direct product of the duplicates, (iii) a baric algebra with weight vector  $\xi^\mu \xi^\nu$ , (iv) a train algebra with principal train roots  $1, 0, \lambda, \mu, \dots$ . These theorems are relevant as follows: (iii) in view of §§ 7, 8; (ii) in view of § 9; (i) in connection with the method used in § 14; (iv) in deriving equations such as (10.10), (12.6).

(Paper VII)

Duplication of an algebra may be compared with the process of forming the second induced matrix of a given matrix (Aitken, 1935; cf. also Wedderburn, 1908 b).

(The marginal figures in red indicate the corresponding equations in Paper V, which are here generalised.)

## § 6. GAMETIC ALGEBRAS.

Consider the inheritance of characters depending on any number of gene differences at any number of loci on any number of chromosomes in a diploid or generally autopolyploid species. Assume that inheritance is symmetrical in the sexes: the sex chromosomes are thus excluded, and crossing over if present must be equal in male and female.

Let  $G^1, G^2, \dots, G^n$  denote the set of gametic types determined by these gene differences. Then there will be

$$m = \frac{1}{2}n(n+1) \quad (6.1)$$

zygotic types  $G^\mu G^\nu (= G^\nu G^\mu)$ . The formulæ giving the series of gametic types produced by each type of individual, and their relative frequencies, may be written

$$G^\mu G^\nu = \sum \gamma_{\sigma}^{\mu\nu} G^\sigma, \quad (6.2) \quad (5.2)$$

with the normalising conditions

$$\sum \gamma_{\sigma}^{\mu\nu} I^{\sigma} = I; \quad (6.3)$$

$\gamma_{\sigma}^{\mu\nu}$  is then the probability that an arbitrary gamete produced by an individual of zygotic type  $G^\mu G^\nu$  is of type  $G^\sigma$ .

(I speak of *zygotic types*—individuals distinguished by the gametes from which they were formed—rather than *genotypes*—individuals distinguished by the gametes which they produce—because the  $G^\mu G^\nu$  are not all distinct genotypes if more than one chromosome is involved: the zygotic algebra, § 7, will have the same train equation if genotypes are used, but will then not be a duplicate algebra.)

A population  $P$  which produces gametes  $G^\mu$  in proportions  $a_\mu$  may be represented by writing

$$P = \sum a_\mu G^\mu. \quad (6.4) \quad (5.4)$$

Imposing the normalising condition

$$\sum a_\mu I^\mu = I, \quad (6.5) \quad (5.7)$$

$a_\mu$  denotes the probability that an arbitrary gamete produced by an arbitrary individual of  $P$  is of type  $G^\mu$ .

A population may also be described by the proportions of the zygotic types  $G^\mu G^\nu$  which it contains; thus we may write

$$P = \sum a_{\mu\nu} G^\mu G^\nu, \quad (6.6) \quad (5.6)$$

with the normalising condition

$$\sum a_{\mu\nu} I^\mu I^\nu = I, \quad (6.7) \quad (5.8)$$

and a similar probability interpretation. We may suppose without loss of generality that  $a_{\mu\nu} = a_{\nu\mu}$ , so that in (6.6) the coefficient of  $G^\mu G^\nu$  is

*J. Paper V,  
pp. 39-40.*



$2a_{\mu\nu}$  if  $\mu \neq \nu$ . The two representations are connected by the gametic series formulæ (6.2); that is to say, from the zygotic representation (6.6) follows the gametic representation

$$P = \sum a_{\mu\nu} \gamma_{\sigma}^{\mu\nu} G^{\sigma}. \quad (6.8)$$

(5.9)

If two populations P, Q intermate at random, representations of the first filial generation are obtained by multiplying the gametic representations of P and Q; i.e. if

$$P = \sum a_{\mu} G^{\mu}, \quad Q = \sum \beta_{\mu} G^{\mu}, \quad (6.9)$$

the population of offspring is

$$PQ = \sum a_{\mu} \beta_{\nu} G^{\mu} G^{\nu} \quad (6.10)$$

$$= \sum a_{\mu} \beta_{\nu} \gamma_{\sigma}^{\mu\nu} G^{\sigma}. \quad (6.11)$$

In particular, the population of offspring of random mating of P within itself is given by  $P^2$ .

We may now view the situation abstractly. The gametic series (6.2) form the multiplication table of a commutative non-associative linear algebra with basis  $G^{\mu} (\mu = 1, \dots, n)$ . It will be called the *gametic algebra* for the type of inheritance considered, and denoted  $\mathbf{G}$ . The equations (6.3) show that  $\mathbf{G}$  is a baric algebra with weight vector

$$\xi^{\mu} = 1^{\mu}. \quad (6.12)$$

With regard to its gametic type frequencies, a population is represented by a normalised element (6.4) of  $\mathbf{G}$ . Multiplication in  $\mathbf{G}$  has the significance described in § 1, and it follows from the multiplicative property of the weight in a baric algebra that PQ will be automatically normalised if P and Q are.

## § 7. ZYGOTIC ALGEBRAS.

When individuals of types  $G^{\mu} G^{\nu}$ ,  $G^{\theta} G^{\phi}$  mate, the probability distribution of zygotic types in their offspring can be obtained by multiplying the gametic representations (given by (6.2)) together, and leaving the product in quadratic form (as in (6.10)). We obtain

$$G^{\mu} G^{\nu} \cdot G^{\theta} G^{\phi} = \sum \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\theta\phi} G^{\sigma} G^{\tau}; \quad (5.10)$$

or, writing

$$Z^{\mu\nu} = G^{\mu} G^{\nu} \quad (7.1) \quad (5.1)$$

to emphasise the union of paired gametes into single individuals,

$$Z^{\mu\nu} Z^{\theta\phi} = \sum \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\theta\phi} Z^{\sigma\tau}. \quad (7.2) \quad (5.3)$$

These  $\frac{1}{2}m(m+1)$  equations, then, are the formulæ giving the series of zygotic types produced by the mating type or *couple*  $Z^{\mu\nu} \times Z^{\theta\phi}$ , the

probability of  $Z^{\sigma\tau}$  being the corresponding coefficient  $\gamma_{\sigma}^{\mu\nu}\gamma_{\tau}^{\theta\phi} + \gamma_{\tau}^{\mu\nu}\gamma_{\sigma}^{\theta\phi}$  (if  $\sigma \neq \tau$ ) or  $\gamma_{\sigma}^{\mu\nu}\gamma_{\sigma}^{\theta\phi}$  (if  $\sigma = \tau$ ).

The linear algebra with basis  $Z^{\mu\nu}$  and multiplication table (7.2) will be called the *zygotic algebra* for the type of inheritance considered. It is a baric algebra with weight vector  $1^{\mu}1^{\nu}$ , the duplicate of the gametic algebra  $\mathbf{G}$ , and will be denoted

$$\mathbf{Z} = \mathbf{G}'. \quad (7.3)$$

A population, regarded as a distribution of zygotic types, is represented by a normalised element

$$P = \sum \alpha_{\mu\nu} Z^{\mu\nu}, \text{ where } \sum \alpha_{\mu\nu} 1^{\mu} 1^{\nu} = 1; \quad (5.5, 8')$$

and multiplication in  $\mathbf{Z}$ , as in  $\mathbf{G}$ , has the significance described in § 1. A product left in quadratic form in the  $Z$ 's gives now the probability distribution of couples  $Z^{\mu\nu}Z^{\theta\phi}$  among the parents; or, as I shall call it, the *copular representation* of the population of offspring.

#### § 8. FURTHER DUPLICATE GENETIC ALGEBRAS.

The process of duplication can be applied repeatedly. Thus the  $\frac{1}{2}m(m+1)$  types of paired zygotes, or couples,

$$K^{\mu\nu, \theta\phi} = Z^{\mu\nu}Z^{\theta\phi}, \quad (8.1) \quad (7.1)$$

can be taken as the basis of a new linear algebra

$$\mathbf{K} = \mathbf{Z}' = \mathbf{G}''. \quad (8.2)$$

Call it the *copular algebra*. A normalised element with positive coefficients

$$P = \sum \alpha_{\mu\nu, \theta\phi} K^{\mu\nu, \theta\phi}, \text{ where } \sum \alpha_{\mu\nu, \theta\phi} 1^{\mu} 1^{\nu} 1^{\theta} 1^{\phi} = 1, \quad (7.4, 5')$$

is the copular representation of a population—the probability distribution of couples in the parents of the individuals comprised in the population.

Similarly, in the next duplicate algebra  $\mathbf{K}'$ , the basic symbols would classify tetrads of grandparents.

In all these algebras, multiplication has the significance described in § 1.

#### § 9. COMBINATION OF GENETIC ALGEBRAS.

Consider two distinct genetic classifications referring to the same population  $P$ , firstly into a set of  $m$  genetic types

$$A^1, A^2, \dots, A^m;$$

secondly into a set of  $n$  genetic types

$$B^1, B^2, \dots, B^n$$

of the same kind (gametic, zygotic, etc.). Let the corresponding genetic algebras be **A**, **B** with multiplication tables

$$A^\mu A^\nu = \sum \gamma_{\sigma}^{\mu\nu} A^\sigma, \quad B^\theta B^\phi = \sum \delta_{\tau}^{\theta\phi} B^\tau. \quad (11.1)$$

By taking account of both classifications at once, we obtain a third classification which may be called their *product*, into  $mn$  genetic types

$$C^{\mu\theta} = A^\mu B^\theta. \quad (11.2)$$

The type  $C^{\mu\theta}$  comprises all individuals (gametes, zygotes, etc.) who are of type  $A^\mu$  in the first classification,  $B^\theta$  in the second.

If the characters of the two classifications are inherited independently, *i.e.* if they involve two quite distinct sets of chromosomes, then the probabilities  $\gamma_{\sigma}^{\mu\nu}$ ,  $\delta_{\tau}^{\theta\phi}$  refer to independent events. Hence the genetic algebra with basis  $C^{\mu\theta}$  is the direct product

$$\mathbf{C} = \mathbf{AB};$$

*i.e.* its multiplication table is

$$C^{\mu\theta} C^{\nu\phi} = \sum \gamma_{\sigma}^{\mu\nu} \delta_{\tau}^{\theta\phi} C^{\sigma\tau}. \quad (11.3)$$

It follows that a genetic algebra which depends on several autosomal linkage groups must be a direct product **ABC** . . . of genetic algebras, one factor algebra for each linkage group.

If, however, the **A** and **B** classifications are independent but genetically linked, *i.e.* if they involve two quite distinct sets of gene loci but not distinct sets of chromosomes, then the probabilities  $\gamma_{\sigma}^{\mu\nu}$ ,  $\delta_{\tau}^{\theta\phi}$  are not independent. Regarded as a linear set, **C** is still the product of the linear sets **A** and **B**; but the algebra **C** will not be the direct product of the algebras **A** and **B** (except in the very exceptional case when all crossing over values between **A** and **B** are precisely 50 per cent.). It is, however, still the case that **C** contains subalgebras isomorphic with **A** and **B**. For example, if these algebras are gametic, and if we keep the first index of  $C^{\mu\theta}$  constant, we are virtually disregarding all the **A**-loci, so we obtain a subalgebra isomorphic with **B**; and this can be done in  $m$  ways.

Hence a genetic algebra based on the allelomorphs of several autosomal loci possesses numerous automorphisms.

It will be shown in § 14 that even when linkage is involved the gametic algebra can be symbolically factorised, and regarded as a symbolic direct product of non-commutative factor algebras, one for each locus (see (14.12)).

#### §§ 10-15. EXAMPLES OF SYMMETRICAL GENETIC ALGEBRAS.

A more detailed description of practical applications will be given elsewhere. My object here is simply to show that the genetic algebras are  
(Paper V)

train algebras. I give in each case the principal and plenary train equations, *i.e.* the identities of lowest degree connecting respectively the sequences of principal and plenary powers of a normalised element. As explained in § 4, these are really recurrence equations, and have a special significance in genetics.

# § 10. SIMPLE MENDELIAN INHERITANCE.

For a single autosomal gene difference (D, R), the gametic multiplication table is

$$DD = D, \quad DR = \frac{1}{2}D + \frac{1}{2}R, \quad RR = R. \quad (10.1)$$

Writing

$$A = DD, \quad B = DR, \quad C = RR, \quad (10.2)$$

we find, *e.g.*,

$$B^2 = (\frac{1}{2}D + \frac{1}{2}R)^2 = \frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C.$$

Hence and similarly the zygotic multiplication table is

$$\begin{aligned} A^2 &= A, & B^2 &= \frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C, & C^2 &= C, \\ BC &= \frac{1}{2}B + \frac{1}{2}C, & CA &= B, & AB &= \frac{1}{2}A + \frac{1}{2}B. \end{aligned} \quad (10.3)$$

Call these two algebras  $G_2, Z_2$  ( $Z_2 = G_2'$ ), and denote their general elements

$$G_2: \quad x = \delta D + \rho R, \quad (10.4)$$

$$Z_2: \quad x = \alpha A + 2\beta B + \gamma C. \quad (10.5)$$

The principal rank equations are

$$G_2: \quad x^2 - (\delta + \rho)x = 0, \quad (10.6)$$

$$Z_2: \quad x^3 - (\alpha + 2\beta + \gamma)x^2 = 0; \quad (10.7)$$

and the plenary rank equations (or identities of lowest degree connecting plenary powers of the general elements) are (10.6) and

$$Z_2: \quad x^{2 \cdot 2} - (\alpha + 2\beta + \gamma)^2 x^2 = 0. \quad (10.8)$$

A population P is represented by an element of unit weight in either algebra, *i.e.* (10.4) or (10.5) with

$$\delta + \rho = 1, \quad \alpha + 2\beta + \gamma = 1,$$

the ratios  $\delta : \rho, \alpha : 2\beta : \gamma$  giving the relative frequencies of the gametic types which it produces or genotypes which it contains. In this case (10.6), (10.7), (10.8) become the train equations

$$G_2: \quad P^2 = P, \quad (10.9)$$

$$Z_2: \quad P^3 = P^2, \quad P^{2 \cdot 2} = P^2, \quad (10.10)$$

expressing facts well known in genetics. It is convenient to write these equations in the form (cf. 4.7)

$$G_2: \quad P(P-1)=0, \quad . \quad . \quad . \quad . \quad . \quad (10.11)$$

$$Z_2: \quad P^2(P-1)=0, \quad P^2[P-1]=0. \quad . \quad . \quad . \quad . \quad (10.12)$$

### § 11. MULTIPLE ALLELOMORPHS.

For  $n$  allelomorphs  $G^\mu (\mu=1, \dots, n)$ , the gametic and zygotic multiplication tables are

$$G^\mu G^\nu = \frac{1}{2}G^\mu + \frac{1}{2}G^\nu, \quad . \quad . \quad . \quad . \quad . \quad (11.1)$$

$$Z^{\mu\nu} Z^{\theta\phi} = \frac{1}{4}Z^{\mu\theta} + \frac{1}{4}Z^{\mu\phi} + \frac{1}{4}Z^{\nu\theta} + \frac{1}{4}Z^{\nu\phi}, \quad . \quad . \quad . \quad . \quad (11.2)$$

where  $Z^{\mu\nu} = G^\mu G^\nu$ . The algebras  $G_n, Z_n$  so determined reduce to  $G_2, Z_2$  when  $n=2$ ; and they have in general the same train equations (10.11), (10.12).

### § 12. LINKED ALLELOMORPHS.

For two linked series of multiple allelomorphs, respectively  $m$  and  $n$  in number, with crossing over probability  $\omega$ , the gametic multiplication table is

$$G^{\mu\alpha} G^{\nu\beta} = \frac{1}{2}(1-\omega)(G^{\mu\alpha} + G^{\nu\beta}) + \frac{1}{2}\omega(G^{\mu\beta} + G^{\nu\alpha}), \quad . \quad . \quad (12.1)$$

where  $G^{\mu\alpha} (\mu=1, \dots, m; \alpha=1, \dots, n)$  are the  $mn$  gametic types. Denote this gametic algebra  $G_{mn}(\omega)$ . The principal and plenary rank equations are

$$x^3 - \frac{1}{2}(3-\omega)\xi x^2 + \frac{1}{2}(1-\omega)\xi^2 x = 0, \quad . \quad . \quad . \quad (12.2)$$

$$x^{2.2} - (2-\omega)\xi^2 x^2 + (1-\omega)\xi^3 x = 0, \quad . \quad . \quad . \quad (12.3)$$

giving for a normalised element  $P$  the train equations

$$P^3 - \frac{1}{2}(3-\omega)P^2 + \frac{1}{2}(1-\omega)P \equiv P(P-1)\left(P - \frac{1-\omega}{2}\right) = 0, \quad . \quad (12.4)$$

$$P^{2.2} - (2-\omega)P^2 + (1-\omega)P \equiv P[P-1][P - (1-\omega)] = 0. \quad . \quad (12.5)$$

In the duplicate algebra  $Z_{mn}(\omega) = G_{mn}(\omega)'$  the corresponding equations are

$$P^2(P-1)\left(P - \frac{1-\omega}{2}\right) = 0, \quad P^2[P-1][P - (1-\omega)] = 0. \quad . \quad (12.6)$$

### § 13. INDEPENDENT ALLELOMORPHS.

Consider two series of multiple allelomorphs in separate autosomal linkage groups. This being indistinguishable from the case of § 12 with  $\omega = \frac{1}{2}$ , the gametic algebra is  $G_{mn}(\frac{1}{2})$ . As in § 9, it may also be expressed as the direct product  $G_m G_n$ .



§ 14. LINKAGE GROUP.

I will first rewrite equations (12.1) with a change of notation. I will then write down the analogous equations for the case of three linked loci, and examine the structure of the corresponding algebra. This will be a sufficient indication of the procedure which can be followed out quite generally for a complete linkage group comprising any number of loci on one autosome, with any number of allelomorphs at each locus. The method may be extended to include any number of linkage groups.

Equations (12.1) may be written

$$AB \cdot A'B' = \frac{1}{2}(1 - \omega)(AB + A'B') + \frac{1}{2}\omega(AB' + A'B). \quad (14.1)$$

Here A and B refer to the two gene loci.  $A^a B^a$  would mean the same as  $G^{aa}$ —a gamete with the  $\mu$ th allelomorph at the A-locus and the  $a$ th at B; but dropping the indices AB and  $A'B'$  stand for any particular gametic types, the same or different.

(14.1) may again be rewritten

$$AB \cdot A'B' = \frac{1}{2}\varpi(A + \chi A')(B + \chi B'), \quad (14.2)$$

where  $\varpi = 1 - \omega$  and  $\chi$  is an operator which interchanges  $\omega$  and  $\varpi$ , so that  $\chi^2 = 1$  and  $\varpi\chi = \omega$ .

Now consider the case of three loci A, B, C, having respectively  $m, n, r$  allelomorphs, and crossing over probabilities  $\omega_{AB}, \omega_{BC}, \omega_{AC}$ . The gametic algebra may be symbolised conveniently as  $G_{mnr}(\omega)$ , where  $\omega$  is the symmetrical matrix of the crossing over values, with diagonal zeros. Its multiplication table, comprising  $\frac{1}{2}mnr(mnr + 1)$  formulæ, is

$$ABC \cdot A'B'C' = \frac{1}{2}\lambda(ABC + A'B'C') + \frac{1}{2}\mu(A'BC + AB'C') + \frac{1}{2}\nu(AB'C + A'BC') + \frac{1}{2}\rho(ABC' + A'B'C), \quad (14.3)$$

where

$$\lambda + \mu + \nu + \rho = 1, \quad (14.4)$$

$$\mu + \nu = \omega_{AB}, \quad \nu + \rho = \omega_{BC}, \quad \mu + \rho = \omega_{AC}. \quad (14.5)$$

The  $\omega$ 's are not independent, but are connected only by an inequality (Haldane, 1918):

$$\omega_{AC} = \omega_{AB} + \omega_{BC} - \kappa\omega_{AB}\omega_{BC}, \quad \text{where } 0 \leq \kappa \leq 2, \quad (14.6)$$

from which may be deduced

$$\mu\rho \geq \nu\lambda. \quad (14.7)$$

Now introduce the following operators:—

$$\left. \begin{array}{lll} \chi_1 \text{ interchanges } \lambda \text{ with } \mu, & \nu \text{ with } \rho, \\ \chi_2 \quad \quad \quad \lambda \quad \quad \nu, & \rho \quad \quad \mu, \\ \chi_3 \quad \quad \quad \lambda \quad \quad \rho, & \mu \quad \quad \nu. \end{array} \right\} \quad (14.8)$$

Together with 1, they form an Abelian group, having the relations

$$\left. \begin{aligned} \chi_2\chi_3 &= \chi_1, & \chi_3\chi_1 &= \chi_2, & \chi_1\chi_2 &= \chi_3, \\ \chi_1^2 &= \chi_2^2 = \chi_3^2 &= \chi_1\chi_2\chi_3 &= 1. \end{aligned} \right\} \quad (14.9)$$

(14.3) may then be rewritten:

$$ABC \cdot A'B'C' = \frac{1}{2}\lambda(A + \chi_1A')(B + \chi_2B')(C + \chi_3C'). \quad (14.10)$$

This symbolism can be manipulated with considerable freedom. For example, an expression such as  $(\alpha ABC + \beta A'BC)$  can be written  $(\alpha A + \beta A')BC$ ; and when two such expressions are multiplied, the distributive law works. The interchange symbols co-operate in the same way.

(14.10) may again be rewritten

$$ABC \cdot A'B'C' = (\chi_0A + \chi_1A')(\chi_0B + \chi_2B')(\chi_0C + \chi_3C'), \quad (14.11)$$

where  $\chi_0 = 1$ , and the operand  $\frac{1}{2}\lambda$  is implied. Finally, (14.11) may be analysed into

$$AA' = \chi_0A + \chi_1A', \quad BB' = \chi_0B + \chi_2B', \quad CC' = \chi_0C + \chi_3C'. \quad (14.12)$$

This separation of the symbols, or factorisation of the algebra (*cf.* end of § 9), will evidently yield valid results, provided that after recombination and application of (14.9),  $\chi_0$  is interpreted as  $\frac{1}{2}\lambda$ ,  $\chi_1$  as  $\frac{1}{2}\mu$ ,  $\chi_2$  as  $\frac{1}{2}\nu$ ,  $\chi_3$  as  $\frac{1}{2}\rho$ . It must be noted that the symbols when separated in this way are non-commutative; *e.g.*  $AA' \neq A'A$ , since  $ABC \cdot A'B'C' \neq A'BC \cdot AB'C'$ .

Select a particular gametic type **ABC**, and write

$$A - A = u, \quad B - B = v, \quad C - C = w, \quad (14.13)$$

where  $A \neq A$ ,  $B \neq B$ ,  $C \neq C$ . Thus the symbols  $u, v, w$  are nil elements having respectively  $m-1, n-1, r-1$  possible values. We have from (14.12):

$$\begin{aligned} A^2 &= (\chi_0 + \chi_1)A, \\ Au &= A^2 - AA = (\chi_0 + \chi_1)A - (\chi_0A + \chi_1A) = \chi_1u, \\ uA &= A^2 - AA = (\chi_0 + \chi_1)A - (\chi_0A + \chi_1A) = \chi_0u, \\ u^2 &= A^2 - AA - AA + A^2 = (\chi_0 + \chi_1)A - (\chi_0A + \chi_1A) - (\chi_0A + \chi_1A) + (\chi_0 + \chi_1)A = 0, \end{aligned}$$

and eight similar equations.

Now write

$$\left. \begin{aligned} ABC &= I, & uBC &= \bar{u}, & AvC &= \bar{v}, & ABw &= \bar{w}, \\ Avw &= \bar{v}\bar{w}, & uBw &= \bar{u}\bar{w}, & uvC &= \bar{u}\bar{v}, & uvw &= \bar{u}\bar{v}\bar{w}. \end{aligned} \right\} \quad (14.14)$$

The symbols  $I, \bar{u}, \bar{v}, \bar{w}, \bar{v}\bar{w}, \bar{w}\bar{u}, \bar{u}\bar{v}, \bar{u}\bar{v}\bar{w}$  thus introduced are linear and linearly independent in the gametic type symbols; and their number is

$$1 + (m-1) + (n-1) + (r-1) + (n-1)(r-1) + (r-1)(m-1) + (m-1)(n-1) + (m-1)(n-1)(r-1) = mn r,$$

which is equal to the number of gametic type symbols. They may thus be taken as a new basis for the gametic algebra. The transformed multiplication table is then easily deduced. We find, for example,

$$I^2 = I, \\ I\bar{u} = A u \cdot B^2 \cdot C^2 = \chi_1(\chi_0 + \chi_2)(\chi_0 + \chi_3)\bar{u} = (\chi_0 + \chi_1 + \chi_2 + \chi_3)\bar{u} = \frac{1}{2}\bar{u},$$

since  $\chi_0 + \chi_1 + \chi_2 + \chi_3$  is to be interpreted as  $\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\rho = \frac{1}{2}$ . Similarly:

$$I\bar{v}\bar{w} = \frac{1}{2}(\lambda + \mu)\bar{v}\bar{w}, \quad I\bar{u}\bar{v}\bar{w} = \frac{1}{2}\lambda\bar{u}\bar{v}\bar{w}, \\ \bar{u}\bar{v} = \frac{1}{2}(\nu + \mu)\bar{u}\bar{v}, \quad \bar{u}\bar{v}\bar{w} = \frac{1}{2}\mu\bar{u}\bar{v}\bar{w}, \quad \bar{u}^2 = \bar{u}\bar{u} = \bar{u}\bar{u}\bar{v}\bar{w} = 0.$$

These results are typical, all other products in the transformed multiplication table being obtainable from them by cyclic permutation of  $u, v, w$  and  $\mu, \nu, \rho$  and 1, 2, 3.

It is now readily verifiable that the algebra has the structure of a special train algebra as defined in § 4, with

$$U = (\bar{u}, \bar{v}, \bar{w}, \bar{v}\bar{w}, \bar{w}\bar{u}, \bar{u}\bar{v}, \bar{u}\bar{v}\bar{w}), \quad U^{(1)} = (\bar{v}\bar{w}, \bar{w}\bar{u}, \bar{u}\bar{v}, \bar{u}\bar{v}\bar{w}), \\ U^{(2)} = (\bar{u}\bar{v}\bar{w}), \quad U^{(3)} = 0.$$

Many of its properties can be most easily deduced from this transformed form. It can be shown that its principal and plenary train roots, other than unity, are the results of

$$\chi_0, \quad \chi_0 + \chi_1, \quad \chi_0 + \chi_2, \quad \chi_0 + \chi_3,$$

operating respectively on  $\frac{1}{2}\lambda$  and  $\lambda$ . Further details are postponed until the properties of special train algebras have been studied elsewhere (*viz.*, in Paper IX, where the analysis of this algebra is continued).

#### § 15. POLYPLOIDY.

A single example—the simplest possible—will illustrate the occurrence of special train algebras in this connection. The gametic algebra with multiplication table

$$\left. \begin{aligned} A^2 &= A, & B^2 &= AC = \frac{1}{6}A + \frac{2}{3}B + \frac{1}{6}C, \\ C^2 &= C, & BC &= \frac{1}{2}B + \frac{1}{2}C, & AB &= \frac{1}{2}A + \frac{1}{2}B, \end{aligned} \right\} \quad (15.1)$$

refers to the inheritance of a single autosomal gene difference in auto-tetraploids. (*Cf.* Haldane, 1930, the case  $m=2$ , with  $A, B, C$  written for  $A^2, Aa, a^2$ .)

This is a special train algebra, as may be seen by performing the transformation

$$A = A, \quad A - B = u, \quad A - 2B + C = p. \quad (15.2)$$

It has the principal and plenary train equations

$$P(P-1)(P-\frac{1}{6})=0, \quad P[P-1][P-\frac{1}{3}]=0. \quad (15.3)$$

## SUMMARY.

A population can be classified genetically at various levels, according to the frequencies of the gametic types which it produces, of the zygotic types of individuals which it contains, of types of mating pairs in the preceding generation, and so on. It is represented accordingly by means of hypercomplex numbers in one or other of a series of linear algebras (gametic, zygotic, copular, . . .), each algebra being isomorphic with the quadratic forms of the preceding algebra. Such a series of *genetic algebras* exists for any mode of genetic inheritance which is symmetrical in the sexes. (Genetic algebras for unsymmetrical inheritance also exist, but are not considered here.) Many calculations which occur in theoretical genetics can be expressed as manipulations within these algebras.

The algebras which arise in this way are all commutative non-associative linear algebras of a special kind. Firstly, they are *baric algebras*, i.e. they possess a scalar representation; secondly, they are *train algebras*, i.e. the rank equation of a suitably normalised hypercomplex number has constant coefficients. Some theorems concerning such algebras are enunciated.

## REFERENCES TO LITERATURE.

- AITKEN, A. C., 1935. "The normal form of compound and induced matrices," *Proc. Lond. Math. Soc.* (2), vol. xxxviii, pp. 354-376.
- DICKSON, L. E., 1914 (reprinted 1930). *Linear algebras*, Cambridge Tract, No. 16.
- ETHERINGTON, I. M. H., 1939. "On non-associative combinations," *Proc. Roy. Soc. Edin.*, vol. lix, pp. 153-162.
- GEPPERT, H., and KOLLER, S., 1938. *Erbmathematik*, Leipzig.
- GLIVENKO, V., 1936. "Algèbre mendélienne," *Moscow Acad. Sci. C.R.*, vol. iv, pp. 385-386.
- HALDANE, J. B. S., 1918. "The combination of linkage values," *Journ. Gen.*, vol. viii, pp. 299-309.
- , 1930. "Theoretical genetics of autopolyploids," *Journ. Gen.*, vol. xxii, pp. 359-372.
- JENNINGS, H. S., 1935. *Genetics*, London.\*
- WEDDERBURN, J. H. M., 1908 (a). "On hypercomplex numbers," *Proc. Lond. Math. Soc.* (2), vol. vi, pp. 77-118.
- , 1908 (b). "On certain theorems in determinants," *Proc. Edin. Math. Soc.* (1), vol. xxvii, pp. 67-69.

(Issued separately November 9, 1939.)

\* The reference in §1 is to the edition of this book published in London. The American edition differs.

PART THREE

NON-ASSOCIATIVE LINEAR ALGEBRAS



V 11

[Proceedings of the Edinburgh Mathematical Society, Series 2, Volume 6, Part iv, Oct. 1941.]

222

## Duplication of linear algebras

By I. M. H. ETHERINGTON.

(Received and read 3rd May, 1940.)

The process of duplication of a linear algebra was defined in an earlier paper<sup>1</sup>, where its occurrence in the symbolism of genetics was pointed out. The definition will now be repeated with an amplification. Although for purpose of illustration it is applied to the algebra of complex numbers, duplication will seem of no special significance if attention is fixed on algebras with associative multiplication and unique division; for duplication generally destroys these properties. The results to be proved, however, show that it is significant in connection with various other conceptions which appeared in the discussion of genetic algebras; namely *baric algebras* and *train algebras* (defined in G.A.), also nilpotent algebras, linear transformation and direct multiplication of algebras.

### §1. Meaning of duplication.

Let  $X$  be a linear algebra of order  $n$  over the field  $F$ , with basis  $a^1, a^2, \dots, a^n$ , having the multiplication table

$$a^\mu a^\nu = \sum_{\sigma=1}^n \gamma_{\sigma}^{\mu\nu} a^\sigma, \quad (\mu, \nu = 1, \dots, n), \quad (\gamma_{\sigma}^{\mu\nu} \in F). \quad (1.1)$$

The commutative and associative laws of multiplication are not assumed. We shall write

$$X = (a^1, a^2, \dots, a^n). \quad (1.2)$$

Except for the positive integer  $n$ , italic letters will be used consistently for hypercomplex numbers, or as they will be called *elements*; and except in the enumerating indices (which always run from 1 to  $n$ ) greek letters (other than  $\Sigma$ ) will be used consistently for elements of  $F$ , which will be called *numbers*. Also  $\Sigma$  will always denote summation with respect to repeated indices. Thus we may without ambiguity

<sup>1</sup> Etherington, "Genetic algebras," *Proc. Roy. Soc., Edin.*, 59 (1939), 242-258. Reference will also be made to "On non-associative combinations," *ibid.*, 153-162. These papers will be referred to as G.A. and N.C. Cf. also *ibid.* (B), 61 (1941), 24-42; *ibid.*

define  $X$  by writing instead of (1.2) and (1.1)

$$X = (a^\mu), \quad (1.3)$$

where

$$a^\mu a^\nu = \sum \gamma_\sigma^{\mu\nu} a^\sigma. \quad (1.4)$$

We have then

$$a^\mu a^\nu \cdot a^\theta a^\phi = \sum \gamma_\sigma^{\mu\nu} a^\sigma \cdot \sum \gamma_\tau^{\theta\phi} a^\tau = \sum \gamma_\sigma^{\mu\nu} \gamma_\tau^{\theta\phi} a^\sigma a^\tau. \quad (1.5)$$

Writing

$$a^\mu a^\nu = a^{\mu\nu}, \quad (1.6)$$

this becomes

$$a^{\mu\nu} a^{\theta\phi} = \sum \gamma_\sigma^{\mu\nu} \gamma_\tau^{\theta\phi} a^{\sigma\tau}, \quad (1.7)$$

which may be regarded as the multiplication table of another linear algebra over the same field  $F$ , denoted

$$X' = (a^{\mu\nu}) \quad (1.8)$$

and called the *duplicate* of  $X$ .

It was assumed in G. A. that  $X$  was commutative; accordingly no distinction was drawn between  $a^{\mu\nu}$  and  $a^{\nu\mu}$ , and  $X'$  was a commutative algebra of order  $\frac{1}{2}n(n+1)$ . It was also pointed out that when  $X$  is non-commutative, the non-commutative algebra  $X'$  is of order  $n^2$ .

In the case when  $X$  is commutative, however, it is still possible in carrying out the process (1.5, 6, 7) to draw a formal distinction between  $a^{\mu\nu}$  and  $a^{\nu\mu}$ , and thus to obtain a *non-commutative duplicate algebra* of order  $n^2$  instead of  $\frac{1}{2}n(n+1)$ . Its multiplication table will still be (1.7), but these equations will now number  $n^4$  instead of

$$\frac{1}{2} \cdot \frac{1}{2}n(n+1) \cdot \{\frac{1}{2}n(n+1) + 1\} = \frac{1}{8}n(n+1)(n^2 + n + 2). \quad (1.9)$$

(Provided that the order of the subalgebra  $X^2$  is not less than 2, multiplication will be non-commutative in the non-commutative duplicate algebra.)

Consider, for example, the algebra of complex numbers,  $Z = (1, i)$  where  $1^2 = 1$ ,  $1i = i1 = i$ ,  $i^2 = -1$ . Its commutative duplicate is  $Z' = (1^2, 1i = i1, i^2)$ , and its non-commutative duplicate is  $Z' = (1^2, 1i, i1, i^2)$ , with multiplication tables respectively

$$\begin{array}{l} 1^2 = a \\ 1i = i1 = b \\ i^2 = c \end{array} \begin{array}{c|cc} a & b & c \\ \hline a & b & -a \\ & c & -b \\ & & a \end{array} \quad (1.10)$$

$$\begin{array}{l} 1^2 = a \\ 1i = b_1 \\ i1 = b_2 \\ i^2 = c \end{array} \begin{array}{c|ccc} a & b_1 & b_2 & c \\ \hline a & b_1 & b_1 & -a \\ b_2 & c & c & -b_2 \\ b_2 & c & c & -b_2 \\ -a & -b_1 & -b_1 & a \end{array} \quad (1.11)$$

These may be contrasted with the "direct square," or direct product of two algebras isomorphic with  $Z$ , say

$$ZZ_1 = (1, i) \times (I, j) = (1I, iI, 1j, ij),$$

having the commutative multiplication table

$$\begin{array}{l}
 1I = a \\
 iI = b_1 \\
 1j = b_2 \\
 ij = c
 \end{array}
 \begin{array}{c}
 a \quad b_1 \quad b_2 \quad c \\
 \hline
 a \quad b_1 \quad b_2 \quad c \\
 -a \quad c \quad -b_2 \\
 -a \quad -b_1 \\
 a
 \end{array}
 \quad (1.12)$$

Like  $Z$ , this is commutative and associative, and possesses a 1-element (having the properties of 1 and  $I$  in the factor algebras), namely  $a$ ; and it has the property of unique division. On the other hand both duplicate algebras are non-associative; and it follows Theorem II (i), (ii), *infra*, that except in the trivial case  $n = 1$  a duplicate algebra cannot be a division algebra or possess a 1-element.

Returning to the general commutative algebra  $X$ , and supposing its order  $> 2$ , we can if desired draw distinctions between  $a^{\mu\nu}$  and  $a^{\nu\mu}$  in some cases but not all (e.g. regard  $a^{12} = a^{21}$ , but other  $a^{\mu\nu} \neq a^{\nu\mu}$ ), and thus obtain intermediate *part-commutative duplicate algebras*, of any order between  $n^2$  and  $\frac{1}{2}n(n+1)$ .

In the rest of this paper, except where otherwise indicated, it is optional whether we assume that  $X$  is non-commutative, in which case  $X'$  is unique; or that  $X$  is commutative and that one of its duplicates is selected as *the* duplicate and denoted  $X'$ . The meaning of the phrase *quadratic form* is fixed accordingly: a quadratic form in  $X$  means a linear combination (coefficients in  $F$ ) of those products of base elements which are distinguished as corresponding to the base elements of  $X'$ .

## § 2. General properties of a duplicate algebra.

To any element

$$x' = \sum a_{\mu\nu} a^{\mu\nu} \quad (2.1)$$

of  $X'$ , there corresponds the quadratic form  $\sum a_{\mu\nu} a^{\mu} a^{\nu}$  in  $X$ . The element  $x'$  and the quadratic form will be called *isomorphs* of each other. The correspondence is unique both ways, and under it the operations of addition and multiplication both hypercomplex and scalar are conserved.

Also, in virtue of (1.6) and the multiplication table (1.4), to any element (2.1) of  $X'$ , there corresponds a unique element of  $X$ , called the *homomorph* of  $x'$ . Again, under this correspondence addition and both kinds of multiplication are conserved; but the correspondence is not unique in the opposite direction. It is nevertheless sometimes convenient (especially in the genetical symbolism) to use " $=$ " for both correspondences, and thus to write:

$$\sum \alpha_{\mu\nu} a^{\mu\nu} = \sum \alpha_{\mu\nu} a^{\mu} a^{\nu} \text{ (its isomorph)} \quad (2.2)$$

$$= \sum \alpha_{\mu\nu} \gamma_{\sigma}^{\mu\nu} a^{\sigma} \text{ (its homomorph).} \quad (2.3)$$

Not all elements of  $X$  are homomorphs: in order that  $x$  should be a homomorph, it is necessary and sufficient that it should be a linear combination of the elements  $\sum \gamma_{\sigma}^{\mu\nu} a^{\sigma}$ , i.e. of  $a^{\mu} a^{\nu}$ ; in other words it must belong to the invariant subalgebra  $X^2$ . Thus the homomorphism is a mapping of  $X'$  on  $X^2$ .

When forming a product in  $X'$ , we may replace the elements to be multiplied by their homomorphs in  $X$ , and then multiply, leaving the product in quadratic form and taking its isomorph in  $X'$ . In symbols, if

$$x' = \sum \alpha_{\mu\nu} a^{\mu\nu}, \quad y' = \sum \beta_{\theta\phi} a^{\theta\phi},$$

then

$$x'y' = \sum \alpha_{\mu\nu} \gamma_{\sigma}^{\mu\nu} a^{\sigma} \cdot \sum \beta_{\theta\phi} \gamma_{\tau}^{\theta\phi} a^{\tau} = \sum \alpha_{\mu\nu} \beta_{\theta\phi} \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\theta\phi} a^{\sigma\tau},$$

which is evidently the correct result. We deduce immediately<sup>1</sup>

**THEOREM I.** *In forming any product, power or continued product in  $X'$ , we can perform all the operations on the homomorphs in  $X$ , only in the final multiplication leaving the product in quadratic form: its isomorph in  $X'$  will then be the result required. (The operations have to be performed in a definite order since multiplication is non-associative.)*

Elements of  $X'$  whose homomorphs are zero will be called *o-elements*; they form a linear set which will be denoted  $O$ .

**THEOREM II.** (i) *Assuming  $n > 1$ ,  $X'$  necessarily contains o-elements other than zero, so that  $O \neq 0$ .* (ii) *In  $X'$ , any product which contains an o-element as one factor is zero.* (iii)  *$O$  is an invariant subalgebra of  $X'$ .* (iv) *The difference algebra  $(X' - O)$  is isomorphic with  $X^2$ .*

<sup>1</sup> In the genetical symbolism, this theorem corresponds to the fact that in order to obtain the distribution of zygotic types of an  $r^{\text{th}}$  filial generation, provided that no selection acts on the zygotes, it is sufficient to trace only the gametic distribution through the  $r-1$  intervening generations.

For if  $n > 1$ , the elements  $\Sigma \gamma_{\sigma}^{\mu\nu} a^{\sigma}$  (i.e.  $a^{\mu} a^{\nu}$ ) are more than  $n$  in number, and therefore cannot be linearly independent; this is equivalent to the statement (i). (ii) follows from Theorem I, and (iii) is an immediate consequence. Or (iii) and (iv) together follow from the general properties of homomorphisms<sup>1</sup>.

In the algebras  $Z'$  of § 1, write

$$a + c = o, \quad b_1 - b_2 = o'; \quad (2.4)$$

these are  $o$ -elements. The multiplication tables (1.10, 11) become

|     | $a$ | $b$     | $o$ |
|-----|-----|---------|-----|
| $a$ | $a$ | $b$     | $0$ |
| $b$ |     | $o - a$ | $0$ |
| $o$ |     |         | $0$ |

(2.5)

|       | $a$        | $b_1$   | $o$ | $o'$ |
|-------|------------|---------|-----|------|
| $a$   | $a$        | $b_1$   | $0$ | $0$  |
| $b_1$ | $b_1 - o'$ | $o - a$ | $0$ | $0$  |
| $o$   | $0$        | $0$     | $0$ | $0$  |
| $o'$  | $0$        | $0$     | $0$ | $0$  |

(2.6)

The zeros in the tables illustrate Theorem II (ii), (iii); while the results of suppressing all the  $o$ 's illustrate the isomorphism of  $(Z' - O)$  with  $Z^2$ , i.e. with  $Z$ .

By a *polynomial* in an element  $x$ , we shall mean a finite linear combination of powers of  $x$ , with coefficients in  $F$ . Since  $X$  does not in general contain a 1-element to serve as an interpretation of  $x^0$  (and even if  $X$  does,  $X'$  does not), we shall exclude from consideration polynomials with a constant term. Thus when multiplication is (a) associative, (b) commutative and non-associative, (c) non-commutative and non-associative, a polynomial means a finite expression (for the index notation see N.C., § 2)

$$\left. \begin{aligned} (a) \quad & ax + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5 + \dots, \\ (b) \quad & ax + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^{2 \cdot 2} + \zeta x^5 + \dots, \\ (c) \quad & ax + \beta x^2 + \gamma x^{2+1} + \delta x^{1+2} + \epsilon x^{(2+1)+1} + \zeta x^{(1+2)+1} + \dots \end{aligned} \right\} \quad (2.7)$$

If  $x$  is the homomorph of  $x'$ , then a polynomial in  $x'$  has as homomorph the same polynomial in  $x$  (perhaps compressed, if multiplication is associative or commutative in  $X$  and not in  $X'$ ).

<sup>1</sup> van der Waerden, *Moderne Algebra* (Berlin, 1930), I, pp. 56-57, where, since the postulate of associative multiplication in rings is not used, the results apply to non-associative algebras. "Invariant subalgebra" is here called *Ideal*, and "difference algebra" *Restklassenring*.



Suppose that every element  $x = \sum a_\sigma \alpha^\sigma$  of  $X$  satisfies the identity

$$f(x, a_\sigma) = 0, \quad (2.8)$$

where  $f(x, a_\sigma)$  is a polynomial in  $x$  whose coefficients are functions of the coordinates  $a_\sigma$  of  $x$ . Then the function  $f(x', \sum a_{\mu\nu} \gamma_{\sigma}^{\mu\nu})$ , formed from any element  $x'$  of  $X'$  and the coordinates of its homomorph in the same way as  $f(x, a_\sigma)$  is formed from  $x$  and its coordinates, is an element of  $X'$  whose homomorph is zero. Hence by Theorem II (ii),

$$x' \cdot f(x', \sum a_{\mu\nu} \gamma_{\sigma}^{\mu\nu}) = 0, \quad f(x', \sum a_{\mu\nu} \gamma_{\sigma}^{\mu\nu}) \cdot x' = 0. \quad (2.9)$$

Thus we have

**THEOREM III.** *If every element  $x = \sum a_\sigma \alpha^\sigma$  of  $X$  satisfies an identity (2.8), then every element  $x' = \sum a_{\mu\nu} \alpha^{\mu\nu}$  of  $X'$  satisfies the identities (2.9). If multiplication is associative or commutative in  $X$  and not in  $X'$ , the function  $f$  in (2.9) may be interpretable in various ways.*

For example, every element  $z = \alpha 1 + \beta i$  of  $Z$  satisfies the rank equation

$$z^2 - 2az + a^2 + \beta^2 = 0.$$

Theorem III can be applied not to this identity but to

$$z^3 - 2az^2 + (a^2 + \beta^2)z = 0.$$

Using the notation (2.4), any element  $z' = \alpha a + \beta b + \gamma o$  of the commutative duplicate algebra has the homomorph  $\alpha 1 + \beta i$ , and therefore satisfies

$$z'^4 - 2az'^3 + (a^2 + \beta^2)z'^2 = 0,$$

which is in fact the rank equation of  $Z'$ . Similarly the element  $z' = \alpha a + \beta b + \gamma o + \delta o'$  of the non-commutative duplicate satisfies the left and right rank equations

$$z'^{1+(1+2)} - 2az'^{1+2} + (a^2 + \beta^2)z'^2 = 0,$$

$$z'^{(2+1)+1} - 2az'^{2+1} + (a^2 + \beta^2)z'^2 = 0;$$

and also satisfies

$$z'^{1+(2+1)} - 2az'^{1+2} + (a^2 + \beta^2)z'^2 = 0,$$

$$z'^{(1+2)+1} - 2az'^{2+1} + (a^2 + \beta^2)z'^2 = 0.$$

### § 3. Related algebras duplicated.

**THEOREM IV.** *If part-commutative duplicate algebras are excluded, the duplicate of a linear transform of  $X$  is a linear transform of  $X'$ .*

For if the equations of transformation of  $X$  are

$$b^\alpha = \sum \lambda_\mu^\alpha a^\mu, \quad a^\mu = \sum \Lambda_\alpha^\mu b^\alpha, \quad (3.1)$$

the multiplication table (1.4) becomes

$$b^\alpha b^\beta = \sum \lambda_\mu^\alpha \lambda_\nu^\beta \gamma_{\sigma}^{\mu\nu} \Lambda_\epsilon^\sigma b^\epsilon. \quad (3.2)$$

Duplicating (commutatively or non-commutatively), we obtain

$$\begin{aligned} b^{a\beta} b^{\gamma\delta} &= \Sigma (\lambda_{\mu}^a \lambda_{\nu}^{\beta} \gamma_{\sigma}^{\mu\nu} \Lambda_{\epsilon}^{\sigma}) (\lambda_{\theta}^{\gamma} \lambda_{\phi}^{\delta} \gamma_{\tau}^{\theta\phi} \Lambda_{\xi}^{\tau}) b^{\epsilon\xi} \\ &= \Sigma (\lambda_{\mu}^a \lambda_{\nu}^{\beta}) (\lambda_{\theta}^{\gamma} \lambda_{\phi}^{\delta}) \gamma_{\sigma}^{\mu\nu} \gamma_{\tau}^{\theta\phi} (\Lambda_{\epsilon}^{\sigma} \Lambda_{\xi}^{\tau} b^{\epsilon\xi}); \end{aligned} \quad (3.3)$$

and this is precisely the result which would be obtained by applying to the duplicate multiplication table (1.7) (commutative or non-commutative correspondingly) the transformation

$$b^{a\beta} = \Sigma \lambda_{\mu}^a \lambda_{\nu}^{\beta} a^{\mu\nu}, \quad a^{\mu\nu} = \Sigma \Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} b^{a\beta}. \quad (3.4)$$

If  $X'$  is (a) a commutative duplicate algebra, or (b) a non-commutative duplicate algebra, it will be seen<sup>1</sup> that the matrix of the induced transformation (3.4) in either direction is (a) the Schläflian (or second induced matrix), or (b) the direct square (or second Burnside matrix), of the matrix of the original transformation (3.1).

Similarly it is easy to prove

**THEOREM V.** *The commutative or non-commutative duplicate of the direct product of two algebras coincides with the direct product of their commutative or non-commutative duplicates. Conversely, the direct product of any two duplicate algebras coincides with a duplicate of their direct product.*

#### § 4. Algebras of special type duplicated.

##### (a) Nilpotent algebras<sup>2</sup>.

Suppose that  $X$  is nilpotent of degree<sup>3</sup>  $2\delta$ ; i.e. in  $X$  all products of  $2\delta$  factors vanish. It will be shown that  $X'$  is nilpotent of degree  $2\delta - 1$ .

<sup>1</sup> See, e.g., Aitken, *Proc. London Math. Soc.* (2), 38 (1935), 354-376.

<sup>2</sup> In this section, as in N.C. § 3,  $\delta, a$  denote positive integers.

<sup>3</sup> *Index* is the usual word in this context - cf. Wedderburn, *Proc. London Math. Soc.* (2), 6 (1908), 77-118; p. 111. But having drawn a distinction in N.C. between *index* and *degree*, I find the latter word more appropriate here. It is perhaps not irrelevant to point out an error in Wedderburn's paper, concerning nilpotent non-associative algebras. It is stated (*loc. cit.*, p. 111) that the sum of all the  $r^{\text{th}}$  powers of such an algebra is less than (i.e. is contained in but is not equal to) the sum of the  $(r-1)^{\text{th}}$  powers. This is not true of the commutative algebra  $X = (a, b, c)$  where  $a^2 = b, ab = b^2 = c, ac = bc = c^2 = 0$ ; for which  $X^2 = (b, c), X^3 = (c), X^4 = 0, X^{2.2} = (c), X^5 = X^{2.2+1} = X^{3+2} = 0$ . For  $X$  is nilpotent of degree 5, whereas  $X^4 + X^{2.2} = X^3$ . Cf. Etherington, "Special train algebras," *Quart. Journ. Math.* (in press). = *Paper IX*, § 2.

Consider first a product  $a'$  in  $X'$  containing  $\delta$  factors. It is a linear combination of products of  $\delta$  base elements. Each product of  $\delta$  base elements is isomorphic with a product of  $2\delta$  base elements in  $X$ , and is therefore an  $o$ -element. Now consider a product  $x'$  of  $2\delta - 1$  factors in  $X'$ : it is expressible as  $a' b'$ , where either  $a'$  or  $b'$  contains at least  $\delta$  factors; i.e. either  $a'$  or  $b'$  is an  $o$ -element, and hence (Theorem II)  $x' = 0$ ; as was to be proved. The same argument applies *a fortiori* if  $X$  is nilpotent of degree  $2\delta - 1$ .

Or suppose (cf. N.C., p. 156) that  $X$  is nilpotent of altitude  $a$ ; i.e. in  $X$  all products of altitude  $a$  vanish. A product  $a'$  in  $X'$  of altitude  $a - 1$  is a linear combination of products of base elements having the same altitude, each isomorphic with a product of altitude  $a$  in the base elements of  $X$ ; and is therefore an  $o$ -element. A product  $x'$  of altitude  $a$  is expressible as  $a' b'$ , where either  $a'$  or  $b'$  is of altitude  $a - 1$  and is thus an  $o$ -element; so that  $x' = 0$ .

We have thus proved:

**THEOREM VI.** *If  $X$  is nilpotent (i) of degree  $2\delta - 1$  or  $2\delta$ , or (ii) of altitude  $a$ , then  $X'$  is nilpotent (i) of degree  $2\delta - 1$ , or (ii) of altitude  $a$ , accordingly.*

(b) *Baric and train algebras.*

If  $X$  is a baric algebra (G.A., § 3), there exists for any element  $x$  a number  $\xi(x)$ , the weight of  $x$ , such that

$$\xi(x + y) = \xi(x) + \xi(y), \quad \xi(xy) = \xi(x) \xi(y), \quad \xi(ax) = a\xi(x).$$

If  $x'$  is any element of  $X'$ , with homomorph  $x$ , and we define  $\xi(x')$  as being equal to  $\xi(x)$ , then it follows that

$$\xi(x' + y') = \xi(x') + \xi(y'), \quad \xi(x' y') = \xi(x') \xi(y'), \quad \xi(ax') = a\xi(x');$$

so that  $\xi(x')$  is a weight function of  $X'$ . Moreover, if

$$\xi(a^\mu) = \xi^\mu,$$

then

$$\xi(a^{\mu\nu}) = \xi(a^\mu a^\nu) = \xi^\mu \xi^\nu.$$

Thus we have

**THEOREM VII.** *If  $X$  is a baric algebra with weight vector  $\xi^\mu$ , then  $X'$  is a baric algebra with weight vector  $\xi^\mu \xi^\nu$ , and the weight of any element in  $X'$  is equal to the weight of its homomorph in  $X$ .*

Combining this with Theorem III, we obtain

**THEOREM VIII.** *If  $X$  is a train algebra with (left or right) principal train roots  $1, \lambda, \mu, \dots$ , then  $X'$  is a train algebra with (left or right) principal train roots included in  $1, 0, \lambda, \mu, \dots$ . Instances of this theorem were observed in G.A.*

It may be stated that the duplicate of a special train algebra (G.A., p. 246), although a train algebra, is not always a special train algebra. The question, which was left open in G.A., whether a train algebra is necessarily a special train algebra, is thus to be answered in the negative.

# COMMUTATIVE TRAIN ALGEBRAS OF RANKS 2 AND 3

I. M. H. ETHERINGTON\*.

[Extracted from the *Journal of the London Mathematical Society*, Vol. 15, 1940.]

## 1. Definitions and notation.

A linear algebra  $\mathbf{X}$  over a field  $\mathbf{F}$  is a *baric algebra* if it possesses a non-trivial scalar representation on  $\mathbf{F}$ . This implies that to any "element"  $x$  of  $\mathbf{X}$  there corresponds a "number"  $\xi(x)$  of  $\mathbf{F}$ , known as the *weight* of  $x$ , not identically zero, such that

$$\xi(x+y) = \xi(x) + \xi(y), \quad \xi(ax) = a\xi(x), \quad \xi(xy) = \xi(x)\xi(y)$$

$$(x, y < \mathbf{X}; a < \mathbf{F}).$$

If, further, the coefficients of the rank equation of  $\mathbf{X}$ , in so far as they depend on the general element  $x$ , depend only on  $\xi(x)$ , then the baric algebra is a *train algebra*. For the general element of unit weight, the rank equation then has constant coefficients, and this is the (principal) *train equation* of the algebra. (In case multiplication is non-commutative, it may be necessary to distinguish *left* and *right* train algebras and equations.)

---

\* Received 10 October, 1939; read 15 December, 1939.

These definitions were given in a previous paper\* to which I shall refer as G.A., and I have shown that such algebras arise naturally in the symbolism of genetics. The commutative and associative laws of multiplication are not assumed in the definitions, and the genetic algebras are, in fact, non-associative though commutative. From §4 onwards the commutative law of multiplication is assumed.

I follow the notation and nomenclature of G.A. without further explanation. Concerning non-associative products and powers, see also an earlier paper†, to which I refer as N.C. For the general theory of linear algebras, reference may be made to the memoirs quoted below‡.

The notation of symbolic powers (G.A., §4) with square brackets is used here only for plenary powers. In interpreting a product involving factors of the form  $x+a$ , it is to be supposed that the expression is first expanded as in ordinary algebra (the factors are therefore commutative and associative); then powers and products are to be interpreted as principal powers if round brackets have been used (e.g., 7.1), as plenary powers ( $x^{[n]} = x^{2^n-1}$ ) if square brackets have been used (e.g. 7.7), as terms of the *operational sequence*

$$x\{y^{n-1}\} = xy.y:y \dots$$

if curly brackets have been used (e.g. 7.10). Genetic interpretations of these three sequences have been given (G.A. §4).

The identities of lowest degree in the three sequences are respectively the *principal*, *plenary*, and *operational rank equations* of an algebra; for normalized elements they may give rise to corresponding *principal*, *plenary*, and *operational train equations*. The *rank* of an algebra means the degree of its (principal) rank equation; one could speak also of *plenary* or *operational rank*.

## 2. Summary.

Some simple properties of baric algebras have already been established. Some further preliminary theorems are given in §§3-5. They are mainly concerned with the invariant subalgebras denoted by **U**, **P**, and the subsets

\* I. M. H. Etherington, "Genetic algebras", *Proc. Royal Soc. Edinburgh*, 59 (1939), 242-258.

† I. M. H. Etherington, "On non-associative combinations", *Proc. Royal Soc. Edinburgh*, 59 (1939), 153-162.

‡ J. H. M. Wedderburn, *Proc. London Math. Soc.* (2), 6 (1908), 77-118. L. E. Dickson, *Algebras and their arithmetics* (Chicago, 1923).



$\mathcal{Q}_\lambda$  of a baric algebra  $\mathbf{X}$ . These are contained each in the next:

$$\mathbf{X} > \mathbf{U} \geq \mathbf{P} \geq \mathcal{Q}_\lambda.$$

The nil subalgebra  $\mathbf{U}$  vanishes (i.e. consists of zero) only in the trivial case when  $\mathbf{X}$  is isomorphic with  $\mathbf{F}$ , as for example in the algebra of real numbers.  $\mathbf{X}$  may then be regarded as a train algebra of rank and order 1 with the rank equation

$$x - \xi(x) = 0.$$

The nilproduct subalgebra  $\mathbf{P}$  (assuming now that multiplication is commutative) vanishes if and only if  $\mathbf{X}$  is a train algebra of rank less than or equal to 2. If  $\mathbf{X}$  is of rank 2, the rank equation must be

$$x(x - \xi) = 0.$$

The subset  $\mathcal{Q}_\lambda$  of triple nilproducts can vanish only when  $\mathbf{X}$  is a train algebra of rank less than or equal to 3. If the rank is 3, the number  $\lambda$  must then be a train root, so that the rank equation is

$$x(x - \xi)(x - \lambda\xi) = 0;$$

and conversely, when this is the case,  $\mathcal{Q}_\lambda = 0$ .

The equations (4.8, 9) give the multiplication table of a commutative baric algebra in its most general form.

The conditions (4.13) for idempotence in a train algebra are significant in the genetical applications, since an idempotent element represents a population in equilibrium for random mating. Applied to the genetic algebra  $\mathbf{Z}_2$ , these conditions give the well-known Pearson-Hardy law\* ( $\beta^2 = \alpha\gamma$  in the notation of G.A., §10). (Paper V, equation 6.4.)

In §6 I examine the properties of train algebras of rank 2. By linear transformation the multiplication table can always be taken as (6.5); this will be recognised as the multiplication table of the genetic algebra  $\mathbf{G}_n$  (G.A., §11). It is shown that multiplication, though generally non-associative, is associative for powers†.

§7 deals with train algebras of rank 3, which occur in various contexts in the genetic symbolism. The main results are:

- (i) If the principal train roots are 1,  $\lambda$ , then the plenary train roots are 1,  $2\lambda$ ; and the operational train roots are included in 1,  $\frac{1}{2}$ ,  $\frac{1}{2}\lambda$ .

\* G. H. Hardy, *Science*, 28 (1908), 49-50.

† This is one of the properties of Jordan's "r-number" algebras [P. Jordan, J. v. Neumann and E. Wigner, *Ann. Math.*, 35 (1934), 29-64], and some of the properties of r-number algebras follow also here: e.g., III (p. 31, Fundamental Theorem 1)  $a^2b \cdot a = a^2 \cdot ba$ ; and (p. 33), Theorem 3 (here  $\mathbf{N}_0 = 0$ ,  $\mathbf{N}_1 = \mathbf{U}$  Cf. also L. E. Dickson, *Trans. American Math. Soc.*, 13 (1912), 59-73, §5.

(ii) The subalgebra generated by a particular element of the algebra (*i.e.* consisting of polynomials\* in this element) is a train algebra of a particular type, having  $\mathbf{U} = \mathbf{P}$ ; for different elements, other than multiples of an idempotent element, the corresponding subalgebras are isomorphic.

(iii) Multiplication is generally non-associative for powers, but the multiplication of indices is commutative (equation 7.21), and hence (*vide* N.C., § 2) distributive both ways. It follows that indices (*i.e.* the shapes of powers) follow the ordinary laws of algebra, except only that the addition of indices is non-associative.

(iv) With the restriction  $\lambda \neq \frac{1}{2}$ , a canonical form for the multiplication table of the algebra is given without proof.

The idempotent element (7.11) may be interpreted genetically as representing the ultimate state of a population, initially  $X$ , in which mating proceeds at random, when inheritance is such as to be characterized by a train algebra of rank 3.

### 3. Nil elements.

Let  $\mathbf{X}$  be a baric algebra of order  $n$ . The letters  $u, v, w$  will be reserved for *nil elements* (elements of zero weight, also called  $u$ -elements), the letters  $p, q$  being reserved for special kinds of nil elements (§§ 4, 5). The set of all nil elements is denoted by  $\mathbf{U}$ , and is the *nil subalgebra* of  $\mathbf{X}$ . Its order is  $n-1$ .

The weight forms by definition a representation (homomorphism) of  $\mathbf{X}$  on  $\mathbf{F}$ , and  $\mathbf{U}$  corresponds to zero in  $\mathbf{F}$ . Hence†

**THEOREM I.**  $\mathbf{U}$  is an invariant subalgebra of  $\mathbf{X}$ , and the difference algebra  $\mathbf{X}-\mathbf{U}$  is isomorphic with  $\mathbf{F}$ .

It is evident that  $\mathbf{X}$  contains at least one element  $A$  of unit weight, which can be taken as one base element. Any nil base element  $u$  can then be replaced, if desired, by  $A-u$  (of weight 1); and any base element  $a$

\* A polynomial means a linear combination of *non-associative* powers.

† Cf. B. L. van der Waerden, *Moderne Algebra* (Berlin, 1930), I, 56-57, where, since the postulate of associative multiplication in rings is not used, the results apply to non-associative algebras. "Invariant subalgebra" is there called *Ideal*, and "difference algebra" *Restklassenring*.

of weight  $\alpha \neq 0$  can be replaced either by  $a/a$  (of weight 1), or by  $A-a/a$  (of weight 0). Hence we have

THEOREM II. *It can be arranged by a linear transformation that any desired number (at least one) of the base elements of  $\mathbf{X}$  shall be of unit weight, and the rest of zero weight; and then the weight of any element is the sum of the coordinates of the "heavy" base elements.*

#### 4. Nilproducts.

Let  $x, y$  be elements of weights  $\xi, \eta$  in the commutative baric algebra  $\mathbf{X}$ . Their *nilproduct* is defined as the nil element

$$x \cdot y = xy - \frac{1}{2}\xi y - \frac{1}{2}\eta x. \quad (4.1)$$

Evidently

$$x \cdot y = y \cdot x, \quad x \cdot (y \pm z) = x \cdot y \pm x \cdot z, \quad (\alpha x) \cdot (\beta y) = \alpha\beta(x \cdot y). \quad (4.2, 3, 4)$$

The *nilsquare*  $x \cdot x = x^2 - \xi x$  is written  $x''$ . For a normalized element,

$$X'' = X^2 - X = X(X-1). \quad (4.5)$$

For nil elements,  $u \cdot v = uv. \quad (4.6)$

The nilsquares of elements of unit weight, and their linear combinations, will be called *p*-elements; and the letter *p*, with enumerating suffixes when required, will be reserved for such elements. The set of all *p*-elements in  $\mathbf{X}$  is proved below (Th. IV) to be an algebra; it will be called the *nilproduct subalgebra* of  $\mathbf{X}$ , and denoted by  $\mathbf{P}$ .

THEOREM III. *All nilproducts are p-elements.*

For if  $x = \xi X, y = \eta Y, \xi \neq 0, \eta \neq 0,$

then identically (multiplication being commutative)

$$x \cdot y = \frac{1}{2}\xi\eta\{4(\frac{1}{2}X + \frac{1}{2}Y)'' - X'' - Y''\} \in \mathbf{P}. \quad (4.7)$$

Also any *u*-element can be expressed as the difference of two elements of equal non-zero weight; whence with (4.3) the theorem follows also for nilproducts of the form  $x \cdot u$  and  $u \cdot v$ .

COROLLARY.  $\mathbf{U} \supseteq \mathbf{P} \supseteq \mathbf{U}^2.$

THEOREM IV. (i)  $\mathbf{P}$  is an invariant subalgebra of  $\mathbf{X}$  (and of  $\mathbf{U}$ ). For  $xp = x \cdot p + \frac{1}{2}\xi p \in \mathbf{P}.$

(ii) The difference algebra  $\mathbf{X} - \mathbf{P}$  is a train algebra with the train equation  $X(X-1) = 0$ . This follows (cf. Th. I) from the homomorphism

$\mathbf{X} \sim \mathbf{X} - \mathbf{P}$ , the element  $X'' = X(X-1)$  in  $\mathbf{X}$  corresponding to zero in  $\mathbf{X} - \mathbf{P}$ . Similarly, the difference algebra  $\mathbf{V} = \mathbf{U} - \mathbf{P}$  satisfies  $\mathbf{V}^2 = 0$ .

THEOREM V. The multiplication table of a commutative baric algebra must always be of the form

$$a^\mu a^\nu = \frac{1}{2} \xi^\mu a^\nu + \frac{1}{2} \xi^\nu a^\mu + \sum \omega_\sigma^{\mu\nu} a^\sigma, \quad (\omega_\sigma^{\mu\nu} = \omega_\sigma^{\nu\mu}), \quad (4.8)$$

where  $\xi^\mu$  are the weights of the base elements  $a^\mu$ , and  $\sum \omega_\sigma^{\mu\nu} a^\sigma$  are  $p$ -elements, namely  $a^\mu a^\nu$ ; since the latter are of zero weight,

$$\sum \omega_\sigma^{\mu\nu} \xi^\sigma = 0. \quad (4.9)$$

Conversely, by the criterion given in G.A., § 3, any multiplication table of the form (4.8), with (4.9) satisfied, determines a commutative baric algebra having the weight vector  $\xi^\mu$ .

The proof is immediate.

Suppose that the element

$$x = \sum \alpha_\sigma a^\sigma \quad (4.10)$$

of the above algebra is idempotent, that is

$$x^2 = x \neq 0,$$

so that all powers of  $x$  are equal. Then its weight  $\xi = \sum \alpha_\sigma \xi^\sigma$  satisfies  $\xi^2 = \xi$  so that

$$\xi = \sum \alpha_\sigma \xi^\sigma = 0 \text{ or } 1. \quad (4.11)$$

Now, from (4.10) with (4.8) and (4.11),

$$\begin{aligned} x^2 &= \frac{1}{2} \sum \alpha_\mu \alpha_\nu (\xi^\mu a^\nu + \xi^\nu a^\mu + 2 \sum \omega_\sigma^{\mu\nu} a^\sigma) \\ &= \frac{1}{2} \sum \alpha_\nu a^\nu \cdot \sum \alpha_\mu \xi^\mu + \frac{1}{2} \sum \alpha_\mu a^\mu \cdot \sum \alpha_\nu \xi^\nu + \sum \alpha_\mu \alpha_\nu \omega_\sigma^{\mu\nu} a^\sigma \\ &= \sum \alpha_\mu \alpha_\nu \omega_\sigma^{\mu\nu} a^\sigma \quad \text{or} \quad x + \sum \alpha_\mu \alpha_\nu \omega_\sigma^{\mu\nu} a^\sigma. \end{aligned}$$

Hence, comparing (4.10), we obtain

THEOREM VI. The conditions for idempotence of the element  $\sum \alpha_\sigma a^\sigma$  are either

$$\text{Case (i):} \quad \sum \alpha_\mu \alpha_\nu \omega_\sigma^{\mu\nu} = \alpha_\sigma \quad (4.12)$$

[which by (4.9) implies  $\sum \alpha_\sigma \xi^\sigma = 0$ ]; or

$$\left. \begin{array}{l} \text{Case (ii):} \quad \sum \alpha_\mu \alpha_\nu \omega_\sigma^{\mu\nu} = 0 \\ \text{together with} \quad \sum \alpha_\sigma \xi^\sigma = 1. \end{array} \right\} \quad (4.13)$$

In a train algebra, Case (i) is clearly impossible; for the rank equation of a nil element takes the form  $u^r = 0$ , so that we cannot have  $u^2 = u \neq 0$ .

### 5. Triple nilproducts.

Let  $\lambda$  be a fixed number of the field  $\mathbf{F}$ , and let  $x, y, z$  be elements of weights  $\xi, \eta, \zeta$  in the commutative baric algebra  $\mathbf{X}$  over  $\mathbf{F}$ . Their *triple nilproduct* is defined as

$$\lambda x \cdot y \cdot z = \frac{1}{3}[x \cdot yz + y \cdot zx + z \cdot xy - (1 + \lambda)(\xi yz + \eta zx + \zeta xy) + \lambda(x\eta\zeta + y\zeta\xi + z\xi\eta)] \quad (5.1)$$

$$= \frac{1}{3}(x - \lambda\xi)(y \cdot z) + \frac{1}{3}(y - \lambda\eta)(z \cdot x) + \frac{1}{3}(z - \lambda\zeta)(x \cdot y), \quad (5.2)$$

abbreviated, when there is no ambiguity, to  $x \cdot y \cdot z$ , and when  $x = y = z$  to  $x^{\cdot\cdot\cdot}$  or  $x^{\cdot\cdot}$  (the *nilcube* of  $x$ ). Evidently

$$x \cdot y \cdot z = x \cdot z \cdot y = \dots, \quad (5.3)$$

i.e., the three factors are commutative and associative;

$$x \cdot y \cdot (z + z') = x \cdot y \cdot z + x \cdot y \cdot z', \dots \quad (5.4)$$

(distributive in each factor);

$$(ax) \cdot (\beta y) \cdot (\gamma z) = a\beta\gamma(x \cdot y \cdot z); \quad (5.5)$$

$$X^{\cdot\cdot\cdot} = X^3 - (1 + \lambda)X^2 + \lambda X = X(X - 1)(X - \lambda) = (X - \lambda)X^{\cdot\cdot}; \quad (5.6)$$

$$u \cdot v \cdot w = \frac{1}{3}(u \cdot vw + v \cdot wu + w \cdot uv). \quad (5.7)$$

The nilcubes of elements of unit weight, and their linear combinations, will be called *q-elements*; and the letter  $q$  will be reserved for them. The set of all  $q$ -elements will be denoted by  $\mathcal{Q}_\lambda$  or  $\mathcal{Q}$ .

**THEOREM VII.** *All triple nilproducts are q-elements.*

In view of (5.4) and (5.5), it is enough to show that any triple nilproduct of the form  $X \cdot Y \cdot Z$  can be expressed as a linear combination of the nilcubes of elements of non-zero weight; and this follows in virtue of the identity

$$6X \cdot Y \cdot Z = (X + Y + Z)^{\cdot\cdot\cdot} - (Y + Z)^{\cdot\cdot\cdot} - (Z + X)^{\cdot\cdot\cdot} - (X + Y)^{\cdot\cdot\cdot} + X^{\cdot\cdot\cdot} + Y^{\cdot\cdot\cdot} + Z^{\cdot\cdot\cdot}. \quad (5.8)$$



In the verification of this identity, which is otherwise straightforward, it is to be noted that, multiplication being commutative and non-associative,

$$\begin{aligned}(X+Y+Z)^3 &= (X+Y+Z)(X^2+Y^2+Z^2+2YZ+2ZX+2XY) \\ &= \Sigma X^3 + \Sigma X^2 Y + 2 \Sigma X \cdot XY + 2 \Sigma X \cdot YZ.\end{aligned}\quad (5.9)$$

THEOREM VIII.  $\mathbf{P} \geq \mathbf{Q}$ .

This follows from (5.6) with Theorem IV (i).

THEOREM IX. If  $\lambda \neq \mu$ , then  $\mathbf{Q}_\lambda + \mathbf{Q}_\mu = \mathbf{P}$ .

For  $(\mu - \lambda)X'' = X''^\lambda - X''^\mu$ , so that  $\mathbf{P} \leq \mathbf{Q}_\lambda + \mathbf{Q}_\mu$ ; and, by Theorem VIII,  $\mathbf{P} \geq \mathbf{Q}_\lambda + \mathbf{Q}_\mu$ .

It is possible in some baric algebras to choose  $\lambda$  in such a way that  $\mathbf{Q}_\lambda$  is an invariant subalgebra of  $\mathbf{X}$ . (This includes cases when  $\mathbf{Q}_\lambda = 0$ .)

THEOREM X. (i) If  $\mathbf{Q}_\lambda$  is an invariant subalgebra of  $\mathbf{X}$ , then the difference algebra  $\mathbf{X} - \mathbf{Q}_\lambda$  is a train algebra with train equation  $X(X-1)(X-\lambda) = 0$ . (For proof, cf. Theorems I and IV.) It follows\* that the rank equation of  $\mathbf{X}$  includes the factor  $x - \lambda\xi$ ; and hence (ii) if further  $\mathbf{X}$  is a train algebra, then  $\lambda$  must be one of its train roots.

#### 6. Commutative train algebras of rank 2.

Let  $\mathbf{X}$  be such an algebra. The train equation must be

$$X(X-1) = 0. \quad (6.1)$$

From (6.1), (4.5) and Theorem III Cor.,

$$\mathbf{P} = \mathbf{U}^2 = 0. \quad (6.2)$$

If therefore  $x, y$  are any two elements of weights  $\xi, \eta$ , since their nilproduct vanishes,

$$xy = \frac{1}{2}\eta x + \frac{1}{2}\xi y; \quad (6.3)$$

in particular

$$uv = 0. \quad (6.4)$$

Hence, taking the base elements to be all of unit weight (Theorem II), say  $I^\mu$  ( $\mu = 1, \dots, n$ ), the multiplication table is

$$I^\mu I^\nu = \frac{1}{2}I^\mu + \frac{1}{2}I^\nu; \quad (6.5)$$

or taking one base element of unit weight and the rest nil, it is

$$I^2 = I, \quad Iu^\sigma = \frac{1}{2}u^\sigma, \quad u^\sigma u^\tau = 0. \quad (6.6)$$

\* Cf. T. Molien *Math. Annalen*, 41 (1893), 83-156, Satz 21.

THEOREM XI. In a commutative train algebra of rank 2, (i) multiplication is associative for powers.

That is, using the notation of N.C.,  $r$  and  $s$  denoting shapes,  $x^r = x^s$  if  $\delta(r) = \delta(s)$ . For from (6.3) we have  $x^2 = \xi x$ ; and hence the value of any power of  $x$  of degree  $n$  is  $\xi^{n-1}x$ . Also, since  $X^2 = X$ , so that all powers of  $X$  are equal,

(ii) any sequence of powers forms a train with the same train equation  $X[X-1] = 0$ .

Also  $XY = \frac{1}{2}X + \frac{1}{2}Y$ , which may be written  $X(Y - \frac{1}{2}) = \frac{1}{2}Y$ . Operating on both sides with  $Y-1$ , we have

$$X\{Y - \frac{1}{2}\}\{Y - 1\} = \frac{1}{2}Y(Y - 1) = 0.$$

Thus (iii) the operational sequence forms a train with the train equation

$$X\{Y - 1\}\{Y - \frac{1}{2}\} = 0.$$

### 7. Commutative train algebras of rank 3.

Let the train equation be

$$X(X-1)(X-\lambda) = 0 = X^3 - (1+\lambda)X^2 + \lambda X. \quad (7.1)$$

The rank equation is then

$$x(x-\xi)(x-\lambda\xi) = 0 = x^3 - (1+\lambda)\xi x^2 + \lambda\xi^2 x. \quad (7.2)$$

Take  $\lambda$  in §5 as the train root  $\lambda$ ; then from (7.1)

$$Q_\lambda = 0, \quad (7.3)$$

and hence all triple nilproducts vanish, e.g.

$$X.YZ + Y.ZX + Z.XY - (1+\lambda)(YZ + ZX + XY) + \lambda(X + Y + Z) = 0, \quad (7.4)$$

$$u.vw + v.wu + w.uv = 0. \quad (7.5)$$

With  $Y = X$ ,  $Z = X^2$ , (7.4) becomes

$$2X^4 + X^{2.2} - (1+\lambda)(2X^3 + X^2) + \lambda(2X + X^2) = 0. \quad (7.6)$$

But from (7.1)

$$2X^4 - 2(1+\lambda)X^3 + 2\lambda X^2 = 0.$$

Hence  $X^{2.2} - (1+2\lambda)X^2 + 2\lambda X = 0$ ; thus we have

THEOREM XII.\* *A commutative train algebra with the principal train equation (7.1) possesses the plenary train equation*

$$X[X-1][X-2\lambda] = 0 = X^{2.2} - (1+2\lambda)X^2 + 2\lambda X. \quad (7.7)$$

The plenary rank equation is obtained by multiplying (7.7) by  $\xi^4$ , and is

$$x^{2.2} - (1+2\lambda)\xi^2 x^2 + 2\lambda\xi^3 x = 0. \quad (7.8)$$

Replacing  $x$  in this equation by  $aX + \beta Y + \gamma Z + \delta W$ , and  $\xi$  by its weight  $a + \beta + \gamma + \delta$ , where  $a, \beta, \gamma, \delta$  are arbitrary, we may equate to zero the coefficient of  $8a\beta\gamma\delta$ . We obtain

$$\begin{aligned} &XY \cdot ZW + XZ \cdot YW + XW \cdot YZ \\ &- (\tfrac{1}{2} + \lambda)(YZ + ZX + XY + XW + YW + ZW) + \tfrac{3}{2}\lambda(X + Y + Z + W) = 0. \end{aligned} \quad (7.9)$$

Now in (7.4) replace  $Z$  by  $Y$  and  $X$  by  $XY$ ; also in (7.9) put

$$W = Z = Y;$$

and eliminate the term  $XY \cdot Y^2$  from the resulting equations. The result is

$$XY \cdot Y : Y - (1 + \lambda)XY \cdot Y + (\tfrac{1}{4} + \lambda)XY - \tfrac{1}{4}\lambda X = \tfrac{1}{4}Y^2 - \tfrac{1}{4}\lambda Y,$$

which may be written in the notation of operational products:

$$X\{Y - \tfrac{1}{2}\}^2\{Y - \lambda\} = \tfrac{1}{4}Y(Y - \lambda).$$

Hence, operating with  $Y - 1$ , we see that the operational sequence  $X\{Y^{n-1}\}$  forms a train satisfying the equation

$$X\{Y - 1\}\{Y - \tfrac{1}{2}\}^2\{Y - \lambda\} = 0. \quad (7.10)$$

In particular algebras, one or both of the factors  $\{Y - \tfrac{1}{2}\}$  may be superfluous: cf. equations (7.15), (7.26).

Further sequences of powers forming trains may be found without difficulty; thus the sequences

$$X, X^3, X^{3.3}, X^{3^3}, \dots,$$

$$X, X^4, X^{4.4}, X^{4^3}, \dots,$$

$$X, X^{1+(2+3)}, X^{[(1+(2+3))+(2+3)], X^{[(1+(2+3))+(2+3))+(2+3)], \dots,$$

form trains with roots respectively

$$1, \lambda + 2\lambda^2; 1, \lambda + \lambda^2 + 2\lambda^3; 1, \lambda;$$

\* The converse can also be proved, that if a commutative basic algebra has the plenary train equation (7.7), then it has the principal train equation (7.1).

the last is a special case of (7.15) *infra*. Such results, which I shall not prove here, can be obtained on the following lines.

A given normalized element  $X$  and its square form the basis of a polynomial subalgebra of  $\mathbf{X}$ , to which all polynomials in  $X$  belong. It will be denoted by  $\mathbf{X}_X$ . By (7.1) and (7.7), it has the commutative multiplication table (7.12). This can be transformed into (7.13), where  $X'' = X^2 - X$ ; or, provided that  $\lambda \neq \frac{1}{2}$ , into (7.14), where

$$I = (X^2 - 2\lambda X)/(1 - 2\lambda). \quad (7.11)$$

$$\begin{array}{cc} & \begin{array}{cc} X & X^2 \end{array} \\ \begin{array}{c} X \\ X^2 \end{array} & \begin{array}{cc} X^2, & (1+\lambda)X^2 - \lambda X \\ & (1+2\lambda)X^2 - 2\lambda X \end{array} \end{array} \quad (7.12)$$

$$\begin{array}{cc} & \begin{array}{cc} X & X'' \end{array} \\ \begin{array}{c} X \\ X'' \end{array} & \begin{array}{cc} X+X'', & \lambda X'' \\ & 0 \end{array} \end{array} \quad (7.13)$$

$$\begin{array}{cc} & \begin{array}{cc} I & X'' \end{array} \\ \begin{array}{c} I \\ X'' \end{array} & \begin{array}{cc} I, & \lambda X'' \\ & 0 \end{array} \end{array} \quad (7.14)$$

We can deduce immediately

**THEOREM XIII.** *The polynomial subalgebras  $\mathbf{X}_X$  generated from different elements  $X$ , of unit weight and not idempotent, are isomorphic, and have all the same principal and plenary train equations as  $\mathbf{X}$ . They have all the operational train equation*

$$Z\{Y-1\}\{Y-\lambda\} = 0. \quad (7.15)$$

*These algebras have the property that their nil and nilproduct subalgebras coincide. If, however,  $X$  is idempotent, then  $X'' = 0$  and  $\mathbf{X}_X$  is isomorphic with  $\mathbf{F}$ .*

Any power of  $X$  is an element of  $\mathbf{X}_X$  and so can be expressed linearly in terms of  $X$  and  $X''$ , i.e. in the form

$$X + (\alpha + \beta\lambda + \gamma\lambda^2 + \delta\lambda^3 + \dots)X''. \quad (7.16)$$

The coefficient of  $X$  in this expression has to be unity, since  $X''$  is of zero weight. Also, for any power other than  $X$  itself,

$$\alpha = 1;$$

for when  $\lambda$  is zero,

$$X^3 = X^{2.2} = X^2 = X + X'',$$

and all higher powers reduce to  $X + X''$ . So the power is characterized by the coefficients  $\beta, \gamma, \delta, \dots$  in (7.16), which may be tabulated as below.

| Power           | $\beta, \gamma, \delta, \dots$ | Power                 | $\beta, \gamma, \delta, \dots$ | Power           | $\beta, \gamma, \delta, \dots$ |
|-----------------|--------------------------------|-----------------------|--------------------------------|-----------------|--------------------------------|
| $X^2$           | 0                              | $X^{2 \cdot 2+1}$     | 1, 2                           | $X^{4+2}$       | 2, 1, 1                        |
| $X^3$           | 1                              | $X^{3+2}$             | 2, 1                           | $X^{2 \cdot 3}$ | 2, 2                           |
| $X^4$           | 1, 1                           | $X^6$                 | 1, 1, 1, 1                     | $X^{3 \cdot 2}$ | 2, 2                           |
| $X^{2 \cdot 2}$ | 2                              | $X^{(2 \cdot 2+1)+1}$ | 1, 1, 2                        | $X^7$           | 1, 1, 1, 1, 1                  |
| $X^5$           | 1, 1, 1                        | $X^{(3+2)+1}$         | 1, 2, 1                        |                 | etc.                           |

These results are obtained in succession, using (7.13). For example, to obtain  $X^{(3+2)2}$  from the results tabulated, we should proceed as follows:

$$\begin{aligned} X^{(3+2)2} &= (X^{3+2})^2 = (X + (1 + 2\lambda + \lambda^2) X'')^2 \\ &= (X + X'') + 2\lambda(1 + 2\lambda + \lambda^2) X'' \\ &= X + (1 + 2\lambda + 4\lambda^2 + 2\lambda^3) X''. \end{aligned}$$

The same result will be found for  $X^{2(3+2)} = X^{2 \cdot 3} X^{2 \cdot 2}$ ; and, generally, if  $r, s$  denote any commutative shapes (*vide* N.C.), then

$$X^{rs} = X^{sr}. \quad (7.17)$$

This is proved as follows.

$$\text{Let } X^r = X + \theta_r X'' \quad (7.18)$$

be the expression of  $X^r$  in the form (7.16). Then

$$\begin{aligned} X^{r \cdot 2} &= (X + \theta_r X'')^2 = (X + X'') + 2\theta_r \lambda X'', \\ (X^r)'' &= X^{r \cdot 2} - X^r = (1 + (2\lambda - 1)\theta_r) X''. \end{aligned} \quad (7.19)$$

Hence, if  $X^s = X + \theta_s X''$ , we get on substituting  $X^r$  for  $X$ ,

$$\begin{aligned} X^{rs} &= X^r + \theta_s (X^r)'' \\ &= (X + \theta_r X'') + \theta_s (1 + (2\lambda - 1)\theta_r) X'' \\ &= X + (\theta_r + \theta_s + (2\lambda - 1)\theta_r \theta_s) X'', \end{aligned} \quad (7.20)$$

which is symmetrical in  $r$  and  $s$ , proving (7.17).

The result is not restricted to elements of unit weight; for we may put  $X = x/\xi$  to unnormalize it, remembering also that for nil elements  $u^3$  and



$u^{2,2}$  and all higher powers vanish in virtue of the rank equations (7.2, 8).

Thus

$$x^{rs} = x^{sr}. \quad (7.21)$$

By the same method of proof, if  $Y(X)$ ,  $Z(X)$  are any polynomials in  $X$ , both of unit weight (i.e. sum of coefficients is unity in each), then

$$Y(Z(X)) = Z(Y(X)). \quad (7.22)$$

This may be unnormalized if the polynomials are homogeneous.

Taking the result (7.21) in conjunction with N.C., §2, we obtain

**THEOREM XIV.** *In a commutative train algebra of rank 3, the indices of powers combine in accordance with the rules*

$$\left. \begin{aligned} a+b &= b+a, & a(b+c) &= ab+ac, & ab \cdot c &= a \cdot bc, \\ ab &= ba, & (b+c)a &= ba+ca, & (a+b)+c &\neq a+(b+c). \end{aligned} \right\} \quad (7.23)$$

The arithmetic of shapes, however, still follows the rules given in N.C. (1).

It remains to examine the multiplication table of  $\mathbf{X}$ , and to obtain if possible a canonical form for it. Suppose that  $\mathbf{X}$  is of order  $1+m+n$ , where  $n$  is the order of  $\mathbf{P}$ . Then the basis of  $\mathbf{X}$  can be chosen as

$$A, \quad u^\sigma \quad (\sigma = 1, \dots, m), \quad p^\alpha \quad (\alpha = 1, \dots, n),$$

where  $A$  is normalized,  $u^\sigma$  are  $u$ -elements but not  $p$ -elements, and  $p^\alpha$  are  $p$ -elements. Then, by Theorem V, we may write

$$\left. \begin{aligned} A^2 &= A + (p), & Au^\sigma &= \frac{1}{2}u^\sigma + (p), & Ap^\alpha &= (p), \\ u^\sigma u^\tau &= (p), & u^\sigma p^\alpha &= (p), & p^\alpha p^\beta &= (p), \end{aligned} \right\} \quad (7.24)$$

where the  $p$ 's in brackets denote unspecified linear combinations of the  $p^\alpha$ . Further conditions, however, are necessary for this to determine a train algebra. I have been able to obtain these conditions completely only with the assumption

$$\lambda \neq \frac{1}{2}.$$

In this case we know that there exists an idempotent element (7.11), which can be used in place of the arbitrary normalized element  $A$ ; and my result (to be proved elsewhere) is that the multiplication table can be reduced by a suitable linear transformation to the form (7.25).

$$\begin{array}{c|ccc} & I & u^\sigma & p^\alpha \\ \hline I & I & \frac{1}{2}u^\sigma & \lambda p^\alpha \\ u^\tau & & \Sigma \gamma_a^{\sigma\tau} p^\alpha & 0 \\ p^\beta & & & 0 \end{array} \quad (7.25)$$

The algebra then has the structure of a special train algebra (G.A., § 4) with

$$\mathbf{U}^{(1)} = \mathbf{P}, \quad \mathbf{U}^{(2)} = \mathbf{Q}_\lambda = 0.$$

It may be shown that the operational train equation is then

$$X\{Y-1\}\{Y-\tfrac{1}{2}\}\{Y-\lambda\} = 0. \quad (7.26)$$

Mathematical Institute,  
University of Edinburgh.

Correction. I hope to prove the above result (7.25) in a paper which I am now writing. I find however that there is a gap in the 'proof' which I had in mind; and it is possible that this multiplication table, though it certainly represents a train algebra of rank 3 with the principal train roots  $1, \lambda$  when  $\lambda \neq \frac{1}{2}$ , is not the most general form possible.

THE QUARTERLY JOURNAL OF  
MATHEMATICS

OXFORD SERIES

Volume 12 No. 45 March 1941

## SPECIAL TRAIN ALGEBRAS

By I. M. H. ETHERINGTON (*Edinburgh*)

[Received 2 July 1940]

(V)

IN a previous paper (4) I have defined three kinds of non-associative linear algebras, namely, *baric algebras*, *train algebras*, and *special train algebras*, which appear in the symbolism of genetics. Train algebras and special train algebras are by definition baric algebras; and I have asserted that special train algebras are train algebras. This paper supplies the proof.

The commutative law of multiplication is assumed in §3 (not in §§1, 2); and the theorem is proved on this assumption in §4, the extension to non-commutative algebras being indicated briefly. Methods of obtaining the principal and plenary train roots of a special train algebra are shown by an example in §5.

## 1. Powers of a non-associative linear algebra

I have defined elsewhere (3) the *shape*  $s$ , *degree*  $\delta$ , and *altitude*  $\alpha$  of a non-associative product or power. (In the case of a power, the *shape* is the same as its *index*.) Of these  $\delta$ ,  $\alpha$  are ordinary integers,  $\delta$  being simply the number of factors; while  $s$  is an integer in an arithmetic with non-associative addition.

By the  $s$ th power of a linear algebra  $A$ , written  $A^s$ , is meant the set of all linear combinations of products having the shape  $s$  formed from elements of  $A$ . By the  $\delta$ th involute of  $A$ , written  $A^{[\delta]}$ , is meant the sum of all powers of  $A$  of degree  $\delta$ ; in other words, the set of all linear combinations of products of  $\delta$  elements of  $A$ .\* By the  $\alpha$ th generation of  $A$ , written  $A^{(\alpha)}$ , is meant the sum of all powers of  $A$  of altitude  $\alpha$ ; in other words, the set of all linear combinations of products of altitude  $\alpha$  formed from elements of  $A$ . Note that

$$A = A^{[1]} = A^{(0)}.$$

By considering how products of a given degree or altitude can be formed from those of lower degree or altitude, we find that

$$A^{[\delta]} = AA^{[\delta-1]} + A^{[2]}A^{[\delta-2]} + \dots + A^{[\delta-1]}A, \quad (1.1)$$

$$A^{(\alpha)} = AA^{(\alpha-1)} + A^{(\alpha-1)}A, \quad (1.2)$$

\* Cf. (8), 111.

equations which can obviously be simplified when multiplication is commutative.

By grouping its factors a non-associative product of given shape can always be regarded as a product of simpler shape, fewer factors, lower altitude. Consequently

$$A^{[\delta]} \leq A^{[\delta-1]} \leq \dots \leq A^{[2]} \leq A, \quad (1.3)$$

$$A^{(\alpha)} \leq A^{(\alpha-1)} \leq \dots \leq A^{(1)} \leq A. \quad (1.4)$$

Since evidently

$$AA^{[\delta]} \leq A^{[\delta]}, \quad A^{[\delta]}A \leq A^{[\delta]}, \quad AA^{(\alpha)} \leq A^{(\alpha)}, \quad A^{(\alpha)}A \leq A^{(\alpha)}, \quad (1.5)$$

it follows that any involute or generation of  $A$  is an invariant sub-algebra of  $A$ . On the other hand, a power of  $A$ ,  $A^s$ , is not necessarily an algebra, since it may not be closed as regards multiplication.

## 2. Nilpotent algebras

A linear algebra  $A$  will be called *nilpotent of degree*  $\delta$  if  $A^{[\delta]} = 0$ ,  $A^{[\delta-1]} \neq 0$ ; and *nilpotent of altitude*  $\alpha$  if  $A^{(\alpha)} = 0$ ,  $A^{(\alpha-1)} \neq 0$ . (Other writers use *index* in place of *degree* in this context.)

In the first case, any product of  $\delta$  or more elements vanishes. Consider a product of altitude  $\alpha' = \delta - 1$ . Its degree  $\delta'$  satisfies  $\delta' \geq \alpha' + 1$ , that is,  $\delta' \geq \delta$ .<sup>\*</sup> Hence this product vanishes, and thus, *if  $A$  is nilpotent of degree  $\delta$ , it is nilpotent of altitude  $\delta - 1$  at most.*

In the second case, any product in  $A$  of altitude  $\alpha$  vanishes. Consider a product of degree  $\delta' = 2^\alpha$ . Its altitude  $\alpha'$  satisfies  $\alpha' \geq \log_2 \delta'$ , that is,  $\alpha' \geq \alpha$ .<sup>\*</sup> Hence this product vanishes; and thus, *if  $A$  is nilpotent of altitude  $\alpha$ , it is nilpotent of degree  $2^\alpha$  at most.*

Wedderburn<sup>†</sup> has stated that, if  $A$  is nilpotent of degree  $\delta$ , then

$$A^{[\delta-1]} < A^{[\delta-2]} < \dots < A^{[2]} < A;$$

but<sup>‡</sup> his proof is based on the incorrect equation<sup>§</sup>

$$A^{[\delta]} = AA^{[\delta-1]} + A^{[\delta-1]}A. \quad (2.1)$$

The same method of proof, however, using (1.2) instead of (2.1), yields the theorem: *If  $A$  is nilpotent of altitude  $\alpha$ , then*

$$A^{(\alpha-1)} < A^{(\alpha-2)} < \dots < A^{(1)} < A. \quad (2.2)$$

## 3. Canonical form of a commutative special train algebra

Let  $X$  be a special train algebra with commutative multiplication. The definition<sup>||</sup> contains three postulates:

<sup>\*</sup> See <sup>II</sup>(3), (14).

<sup>†</sup> (8), 111.

<sup>§</sup> Cf. equation (1.1) here.

<sup>‡</sup> Cf. <sup>VII</sup>(6), § 4, footnote.

<sup>||</sup> <sup>VI</sup>(4), 246.

(1)  $X$  is a baric algebra. Hence\* we can provide by a suitable linear transformation that one base element of  $X$ , say  $A$ , shall be of unit weight, and the rest, say  $u^\sigma$  ( $\sigma = 1, \dots, n$ ), of zero weight. We shall suppose this done. The  $u^\sigma$  form the basis of the nil sub-algebra  $U$ , which we know is an invariant sub-algebra of  $X$ ; and the multiplication table of  $X$  takes the form

$$A^2 = A + (u), \quad Au^\sigma = (u), \quad u^\sigma u^\tau = (u), \quad (3.1)$$

where the  $u$ 's in brackets denote unspecified linear combinations of the  $u^\sigma$ .

(2)  $U$  is nilpotent of a certain degree. As shown in § 2, this is equivalent to saying that  $U$  is nilpotent of a certain altitude, say  $\alpha$ . Then we know that

$$0 = U^{(\alpha)} < U^{(\alpha-1)} < U^{(\alpha-2)} < \dots < U^{(1)} < U. \quad (3.2)$$

Consequently, by an appropriate linear transformation of the  $u^\sigma$ , we can separate the basis of  $U$  into sets of, let us say,  $u_0$ -elements,  $u_1$ -elements,  $u_2$ -elements, etc., such that the  $u_0$ 's belong to  $U$  but not to  $U^{(1)}$ , the  $u_1$ 's belong to  $U^{(1)}$  but not to  $U^{(2)}$ , the  $u_2$ 's belong to  $U^{(2)}$  but not to  $U^{(3)}$ , and so on. Since (cf. (1.2))  $UU^{(\theta)} = U^{(\theta+1)}$ , the multiplication table of  $U$  will take the form

|         | $(u_0)$             | $(u_1)$             | $(u_2)$        | . | . | . | $(u_{\alpha-2})$ | $(u_{\alpha-1})$ |
|---------|---------------------|---------------------|----------------|---|---|---|------------------|------------------|
| $(u_0)$ | $(u_1, u_2, \dots)$ | $(u_2, u_3, \dots)$ | $(u_3, \dots)$ | . | . | . | $(u_{\alpha-1})$ | 0                |
| $(u_1)$ |                     | $(u_2, u_3, \dots)$ | $(u_3, \dots)$ | . | . | . | $(u_{\alpha-1})$ | 0                |
| $(u_2)$ |                     |                     | $(u_3, \dots)$ | . | . | . | $(u_{\alpha-1})$ | 0                |
| .       |                     |                     |                |   |   |   | .                | .                |
| .       |                     |                     |                |   |   |   | .                | .                |
| .       |                     |                     |                |   |   |   | .                | .                |

(3.3)

That is to say, the product of any two  $u_0$ -elements belongs to  $U^{(1)}$  and hence may involve  $u_1$ -elements,  $u_2$ -elements, etc., but not  $u_0$ -elements; the product of a  $u_0$  and a  $u_1$  may involve  $u_2$ 's,  $u_3$ 's, etc., but not  $u_0$ 's or  $u_1$ 's; and so on.

(3)  $U^{(1)}, U^{(2)}, \dots, U^{(\alpha-1)}$  are invariant sub-algebras of  $X$ ; so, of course, is  $U$ . This gives us the further information that

$$A(u_0) = (u_0, u_1, \dots), \quad A(u_1) = (u_1, u_2, \dots), \quad A(u_2) = (u_2, u_3, \dots), \dots \quad (3.4)$$

\* (5), Theorem II.  
VIII



That is to say, the product of  $A$  and any  $u_\theta$  may involve the  $u_\theta$ 's,  $u_{\theta+1}$ 's,  $u_{\theta+2}$ 's, ..., but not the  $u_{\theta-1}$ 's,  $u_{\theta-2}$ 's, ...

Consider now each  $u_\theta$  set of base elements as forming a column vector; and write the multiplication rules (3.4) in the form

$$Au_\theta = P_\theta u_\theta + Q_\theta u_{\theta+1} + R_\theta u_{\theta+2} + \dots, \quad (3.5)$$

where each  $P_\theta$  is a square matrix and  $Q_\theta, R_\theta, \dots$  are in general rectangular matrices.

A linear transformation of  $X$  affecting only the  $u_\theta$ 's, say

$$v_\theta = Hu_\theta, \quad u_\theta = H^{-1}v_\theta, \quad (3.6)$$

transforms (3.5) into

$$Av_\theta = HP_\theta H^{-1}v_\theta + HQ_\theta u_{\theta+1} + HR_\theta u_{\theta+2} + \dots, \quad (3.7)$$

and thus induces a collineatory transformation ( $HP_\theta H^{-1}$ ) of the matrix  $P_\theta$ . Conversely, any such transformation of  $P_\theta$  corresponds to a linear transformation of the  $u_\theta$ 's.

Take  $H$  to be the matrix which reduces  $P_\theta$  to its classical canonical form.\* Let us suppose that the corresponding transformation (3.6) has been carried out, and that this has been done for each of the  $\alpha$  sets of nil base elements. We shall then say that the special train algebra  $X$  has been given its *canonical form*.

In what follows, it is not essential that the above reduction shall have been carried out completely. It is sufficient if each  $P_\theta$  has been reduced to the Jacobian form.† That is to say, the essential thing is that the latent roots of each matrix  $P_\theta$  shall appear down its main diagonal, with zeros below the diagonal.

Let us return to the original notation  $A, u^\sigma$  ( $\sigma = 1, \dots, n$ ) for the basis of  $X$ , without, however, changing the order in which the nil base elements have been placed. The multiplication table of  $X$  has now the form

$$A^2 = A + \dots, \quad Au^\sigma = \lambda_\sigma u^\sigma + \dots, \quad u^\sigma u^\tau = 0.u^\sigma + 0.u^\tau + \dots, \quad (3.8)$$

where in each equation the terms omitted involve only  $u$ 's with higher suffixes than those written down. (The  $\lambda_\sigma$  are the latent roots,  $n$  in all, of the  $\alpha$  matrices  $P_0, P_1, \dots, P_{\alpha-1}$ .) Conversely, it is easy to see that a commutative linear algebra whose multiplication table has this form has the defining properties of a special train algebra.

\* (7), Ch. VI.

† (7), 64.

#### 4. Characteristic and rank equations of a special train algebra

Consider a commutative special train algebra  $X$  in the canonical (or partially reduced) form described above. Let its general element be written as

$$x = \xi A + \sum \alpha_\sigma u^\sigma, \quad (4.1)$$

$\xi$  being its weight. Then, using (3.8),

$$xA = \xi A + \dots, \quad (4.2)$$

$$xu^\tau = \xi \lambda_\tau u^\tau + \dots,$$

so that

$$\left. \begin{aligned} 0 &= (\xi - x)A + \dots \\ 0 &= (\xi \lambda_1 - x)u^1 + \dots \\ 0 &= (\xi \lambda_2 - x)u^2 + \dots \quad \text{etc.} \end{aligned} \right\} \quad (4.3)$$

The characteristic equation\* of the algebra is obtained by equating to zero  $x$  times the determinant of this set of equations; it is therefore

$$x(x - \xi)(x - \lambda_1 \xi) \dots (x - \lambda_n \xi) = 0. \quad (4.4)$$

Powers of  $x$  in the expanded form of this equation are to be interpreted as principal powers.†

If the rank equation of  $X$  is  $f(x) = 0$ , then it is known‡ that  $f(x)$  must be a factor of the left side of the characteristic equation (4.4). The rank equation therefore has the form

$$x(x - \xi)(x - \mu_1 \xi) \dots (x - \mu_r \xi) = 0, \quad (4.5)$$

where  $\mu_1, \dots, \mu_r$  are included in  $\lambda_1, \dots, \lambda_n$ .  $X$  has thus the essential property of a train algebra, and the theorem has been proved.

When multiplication is non-commutative, the third defining postulate of a special train algebra means both

$$XU^{(\theta)} \leq U^{(\theta)} \quad (\theta = 1, \dots, \alpha - 1) \quad (4.6)$$

and

$$U^{(\theta)}X \leq U^{(\theta)} \quad (\theta = 1, \dots, \alpha - 1). \quad (4.7)$$

If only (4.6) is assumed, the algebra may be called a *left special train algebra*; the analysis as given leads to a left rank equation, and we find that  $X$  is a left train algebra. Similarly, assuming only (4.7), the algebra is a *right special train algebra*; and, using products of the type  $uA$  instead of  $Au$ , we can show that  $X$  is a right train algebra. When both (4.6) and (4.7) hold, the two canonical forms will not necessarily coincide; and in fact the left and right ranks need not be equal.

\* (2), § 15, Theorem 3.

†  $\overline{VI}$  (4), § 2.

‡ (1), § 3, Cor. II.

### 5. An example from genetics

In ~~(4)~~ § 14 I considered the genetic algebra for the gametic types depending on three linked series of multiple allelomorphs, having the multiplication table ~~(4)~~ (14.3). The transformation which was applied to this had the effect, it will now be seen, of giving this commutative special train algebra its canonical form. Writing  $u, v, w, p, q, r, s$  for  $\bar{u}, \bar{v}, \bar{w}, \bar{v}\bar{w}, \bar{w}\bar{u}, \bar{u}\bar{v}$ , and  $\theta, \phi, \psi$  for  $1-\omega_{BC}, 1-\omega_{AC}, 1-\omega_{AB}$ , the transformed multiplication table is

|              | $I$ | $u$            | $v$                       | $w$                       | $p$                   | $q$                 | $r$                 | $s$                    |
|--------------|-----|----------------|---------------------------|---------------------------|-----------------------|---------------------|---------------------|------------------------|
| $I$          | $I$ | $\frac{1}{2}u$ | $\frac{1}{2}v$            | $\frac{1}{2}w$            | $\frac{1}{2}\theta p$ | $\frac{1}{2}\phi q$ | $\frac{1}{2}\psi r$ | $\frac{1}{2}\lambda s$ |
| $u$          |     | 0              | $\frac{1}{2}\omega_{AB}r$ | $\frac{1}{2}\omega_{AC}q$ | $\frac{1}{2}\mu s$    | 0                   | 0                   | 0                      |
| $v$          |     |                | 0                         | $\frac{1}{2}\omega_{BC}p$ | 0                     | $\frac{1}{2}\nu s$  | 0                   | 0                      |
| $w$          |     |                |                           | 0                         | 0                     | 0                   | $\frac{1}{2}\rho s$ | 0                      |
| $p, q, r, s$ |     |                |                           |                           | 0                     | 0                   | 0                   | 0                      |

In the notation of § 3,  $u, v, w$  form the  $u_0$  set of base elements;  $p, q, r$  are  $u_1$ -elements; and  $s$  is  $u_2$ . The characteristic equation (4.4) is seen to be

$$x(x-\xi)(x-\frac{1}{2}\xi)^3(x-\frac{1}{2}\theta\xi)(x-\frac{1}{2}\phi\xi)(x-\frac{1}{2}\psi\xi)(x-\frac{1}{2}\lambda\xi) = 0. \quad (5.1)$$

The rank equation is found as follows. Consider the general element

$$x = \xi I + \alpha u + \beta v + \gamma w + \text{terms in } p, q, r, s,$$

whose square is

$$x^2 = \xi^2 I + \xi \alpha u + \xi \beta v + \xi \gamma w + \text{terms in } p, q, r, s.$$

We have

$$x(x-\xi) = x^2 - \xi x = \kappa p + \text{terms in } q, r, s \text{ (say),}$$

$$x(x-\xi)x = \xi \kappa \cdot \frac{1}{2}\theta p + \text{terms in } q, r, s.$$

Hence

$$x(x-\xi)(x-\frac{1}{2}\theta\xi) = \text{terms in } q, r, s.$$

Similarly,

$$x(x-\xi)(x-\frac{1}{2}\theta\xi)(x-\frac{1}{2}\phi\xi) = \text{terms in } r, s,$$

$$x(x-\xi)(x-\frac{1}{2}\theta\xi)(x-\frac{1}{2}\phi\xi)(x-\frac{1}{2}\psi\xi) = \text{a multiple of } s,$$

$$x(x-\xi)(x-\frac{1}{2}\theta\xi)(x-\frac{1}{2}\phi\xi)(x-\frac{1}{2}\psi\xi)(x-\frac{1}{2}\lambda\xi) = 0. \quad (5.2)$$

By considering particular elements such as  $I+u, I+p, I+q, I+r, I+s$ , we can show that no factor in this equation is superfluous, and

it is thus the rank equation of the algebra. The principal train roots are therefore  $1, \frac{1}{2}\theta, \frac{1}{2}\phi, \frac{1}{2}\psi, \frac{1}{2}\lambda$ .

The plenary rank equation,\* or equation of lowest degree connecting plenary powers ( $x^{[m]} = x^{2^{m-1}}$ ) can be obtained most simply by use of *annulling polynomials*, as follows. Consider for simplicity only a normalized element

$$X = I + \alpha u + \beta v + \gamma w + \delta p + \epsilon q + \zeta r + \eta s. \quad (5.3)$$

Let  $\Phi X$  denote  $X^2$ , so that

$$\begin{aligned} \Phi X = I + \alpha u + \beta v + \gamma w + (\delta\theta + \beta\gamma\omega_{BC})p + (\dots)q + (\dots)r + \\ + (\eta\lambda + \alpha\delta\mu + \beta\epsilon\nu + \gamma\zeta\rho)s. \end{aligned} \quad (5.4)$$

The operator  $\Phi$  is to be considered as acting only on the coefficients of  $X$ . Its effect on any one coefficient, of course, depends on the values of the other coefficients as well. Thus

$$\left. \begin{aligned} \Phi\alpha &= \alpha, & \Phi\beta &= \beta, & \Phi\gamma &= \gamma \\ \Phi\delta &= \delta\theta + \beta\gamma\omega_{BC}, \dots, \dots \\ \Phi\eta &= \eta\lambda + \alpha\delta\mu + \beta\epsilon\nu + \gamma\zeta\rho \end{aligned} \right\}. \quad (5.5)$$

Therefore  $(\Phi-1)\alpha = 0$ ; or, as we may say, the polynomial

$$\Phi - 1 \quad (5.6)$$

annuls  $\alpha$ ; it also annuls  $\beta$  and  $\gamma$ . Also  $(\Phi-\theta)\delta = \beta\gamma\omega_{BC}$ , which is annulled by  $\Phi-1$ . Thus, of the polynomials

$$(\Phi-1)(\Phi-\theta), \quad (\Phi-1)(\Phi-\phi), \quad (\Phi-1)(\Phi-\psi), \quad (5.7)$$

the first annuls  $\delta$ , and similarly the other two annul respectively  $\epsilon$  and  $\zeta$ . Also

$$(\Phi-\lambda)\eta = \alpha\delta\mu + \beta\epsilon\nu + \gamma\zeta\rho.$$

Each term on the right is annulled by one of the operators (5.7); hence  $(\Phi-\lambda)\eta$  is annulled by the L.C.M. of these operators, and  $\eta$  by

$$(\Phi-1)(\Phi-\theta)(\Phi-\phi)(\Phi-\psi)(\Phi-\lambda). \quad (5.8)$$

Finally, the annulling polynomial for  $X$  itself is the L.C.M. of the five operators (5.6), (5.7), (5.8). Thus we have the plenary train equation

$$(\Phi-1)(\Phi-\theta)(\Phi-\phi)(\Phi-\psi)(\Phi-\lambda)X = 0, \quad (5.9)$$

$$\text{i.e.} \quad X[X-1][X-\theta][X-\phi][X-\psi][X-\lambda] = 0, \quad (5.10)$$

in which, after expansion, symbolic powers of  $X$  are to be interpreted as plenary powers.

We have shown, then, that the sequence of plenary powers forms a train, and that the plenary train roots are  $1, \theta, \phi, \psi, \lambda$ .

The genetic interpretation of train roots has already been given.†

\* See  $\sqrt[4]{X}$ , 246-7.

†  $\sqrt[4]{X}$ , 247.

*Added 16 December 1940.* It has been tacitly assumed in § 5 that  $\theta, \phi, \psi, 1$  are all unequal. Genetically, we may suppose without loss of generality that the loci  $A, B, C$  are distinct and occur in that order on a chromosome, with

$$0 < \omega_{BC} \leq \omega_{AB} < \omega_{AC} < \frac{1}{2}.$$

Then it may be shown that

$$\frac{1}{4} < \lambda < \phi < \psi \leq \theta < 1.$$

Thus an exception to the tacit assumption occurs when  $\omega_{BC} = \omega_{AB}$ ; in this case  $\psi = \theta$ , the equations (5.2), (5.10) contain repeated factors, and the repetitions are superfluous.

#### REFERENCES

1. L. E. Dickson, 'Linear algebras': *Trans. American Math. Soc.* 13 (1912), 59-73.
2. ——— *Linear Algebras*, Cambridge Tract No. 16, 1914 (reprinted 1930).
3. I. M. H. Etherington, 'On non-associative combinations': *Proc. Roy. Soc. Edinburgh*, 59 (1939), 153-62.
4. ——— 'Genetic algebras': *ibid.* 242-58.
5. ——— 'Commutative train algebras of ranks 2 and 3': *J. of London Math. Soc.* 15 (1940), 136-49.
6. ——— 'Duplication of linear algebras': *Proc. Edinburgh Math. Soc.* (2) 6. (In press).
7. H. W. Turnbull and A. C. Aitken, *Introduction to the Theory of Canonical Matrices*, 1932.
8. J. H. M. Wedderburn, 'On hypercomplex numbers': *Proc. London Math. Soc.* (2) 6 (1908), 77-118.



# POSTSCRIPT TO PAPER IX

A second example from genetics.

It was stated in Paper VI, p. 243, that unsymmetrical genetic algebras possess train subalgebras. I will illustrate this by consideration of the algebra representing sex-linked inheritance given in Paper V, (11.10), whose multiplication table can be written

|                     |                    |                    |  |
|---------------------|--------------------|--------------------|--|
|                     | d                  | e                  |  |
| a                   | $\frac{1}{2}I$     | $\frac{1}{2}J$     | wherein                                      |
| b                   | $\frac{1}{4}(I+K)$ | $\frac{1}{4}(J+L)$ |  |
| c                   | $\frac{1}{2}K$     | $\frac{1}{2}L$     |  |
| other products = 0, |                    |                    |  |
|                     |                    |                    | I = a+d,<br>J = b+d,<br>K = b+e,<br>L = c+e. |

Call the algebra  $\mathcal{X}$  and consider its invariant subalgebra  $\mathcal{X}^2$ , for which, as is clear from the multiplication table, the four linearly independent symbols I, J, K, L form a basis.

The general element of  $\mathcal{X}^2$  may be written

$$x = \lambda I + \mu J + \nu K + \rho L. \quad (1)$$

Expressed in the original notation, this is equal to

$$\alpha a + \beta b + \gamma c + \delta d + \epsilon e$$

where  $\alpha = \lambda$ ,  $\beta = \mu + \nu$ ,  $\gamma = \rho$ ,  $\delta = \lambda + \mu$ ,  $\epsilon = \nu + \rho$ .

Eliminating  $\lambda, \mu, \nu, \rho$  from these equations, we see that the subalgebra  $\mathcal{X}^2$  consists of those elements of  $\mathcal{X}$  for which

$$\alpha + \beta + \gamma = \delta + \epsilon.$$

Now (cf. the normalisation equations (11.12) in Paper V) we are interested genetically in those elements of  $\mathcal{X}$  for which

$$\alpha + \beta + \gamma = \delta + \epsilon = 1; \quad (2)$$

that is to say, in those elements of the subalgebra  $\mathcal{X}^2$  for which

$$\lambda + \mu + \nu + \rho = 1. \quad (3)$$

Write

$$u = 2I - J - K = 2a - 2b + d - e,$$

$$p = I - 2J + K = a - b - d + e,$$

$$q = I - J - K + L = a - 2b + c;$$

and let us take the four linearly independent quantities I, u, p, q instead of I, J, K, L as the basis of  $\mathcal{X}^2$ . It will be found that the multiplication table of  $\mathcal{X}^2$  is then

|   | I | u              | p               | q |
|---|---|----------------|-----------------|---|
| I | I | $\frac{1}{2}u$ | $-\frac{1}{4}p$ | 0 |
| u |   | u              | $-\frac{1}{4}q$ | 0 |
| p |   |                | $-\frac{1}{2}q$ | 0 |
| q |   |                |                 | 0 |

which is the multiplication table of a special train algebra in canonical form, having the principal train equation

$$x^2(x-1)(x+\frac{1}{4})=0.$$

In this train algebra, the symbols I, J, K, L are of unit weight. The weight of the element (1) is therefore  $\lambda + \mu + \nu + \rho$ ; and the normalisation equations (2), which are equivalent to (3), express the fact that (1) is of unit weight. In other words, the elements in which we are interested genetically, representing populations, which are the normalised elements of the original algebra  $\mathcal{X}$ , are the elements of unit weight in the special train subalgebra  $\mathcal{X}^2$ .

The plenary train equation of  $\mathcal{X}^2$  is not quite what one might expect by analogy with VIII, Theorem XII, and with IX (5.10). I will find it by the method used in IX, p. 7. Writing

$$X = I + \alpha u + \beta p + \gamma q,$$

(where  $\alpha, \beta, \gamma$  are not the same as above), we have

$$\Phi X = X^2 = I + \alpha u - \frac{1}{2}\beta p + (\alpha^2 - \frac{1}{2}\alpha\beta - \frac{1}{2}\beta^2)q.$$

Therefore

$$\Phi\alpha = \alpha, \text{ or } \Phi - 1 \text{ annuls } \alpha;$$

$$\Phi\beta = -\frac{1}{2}\beta, \text{ or } \Phi + \frac{1}{2} \text{ annuls } \beta.$$

Also

$$\Phi\gamma = \alpha^2 - \frac{1}{2}\alpha\beta - \frac{1}{2}\beta^2,$$

in which<sup>\*</sup>  $\alpha^2$  is annulled by  $\Phi - 1$ ,

$\alpha\beta$  is annulled by  $\Phi + \frac{1}{2}$ ,

$\beta^2$  is annulled by  $\Phi - \frac{1}{4}$ ,

so that  $\gamma$  is annulled by  $\Phi(\Phi-1)(\Phi+\frac{1}{2})(\Phi-\frac{1}{4})$ ,

and hence so also is X. That is, the plenary train equation is

$$x^2[x-1][x+\frac{1}{2}][x-\frac{1}{4}]=0.$$

<sup>\*</sup>Lemma: If  $\Phi-\lambda$  annuls  $\alpha$ , and  $\Phi-\mu$  annuls  $\beta$ , then

$\Phi\alpha = \lambda\alpha$ ,  $\Phi\beta = \mu\beta$ ,  $\Phi(\alpha\beta) = \lambda\mu\alpha\beta$   
and therefore  $\Phi-\lambda\mu$  annuls  $\alpha\beta$ . Similarly  $\Phi-\lambda^2$ ,  $\Phi-\mu^2$  annuls  $\alpha^2$  and  $\beta^2$ .

X

48

SOME NON-ASSOCIATIVE ALGEBRAS IN WHICH THE  
MULTIPLICATION OF INDICES IS COMMUTATIVE

I. M. H. ETHERINGTON§§.

[*Extracted from the Journal of the London Mathematical Society, Vol. 16, 1941.*]

1. This paper deals with some linear algebras in which multiplication is commutative but non-associative, including non-associative for powers

---

§§ Received 11 November, 1940; read 17 December, 1940.

[e.g.,  $x(x(x^2)) \neq (x^2)^2$ ]; but for which the property

$$x^{rs} = x^{sr}, \quad \text{i.e.} \quad (x^r)^s = (x^s)^r, \quad (1.1)$$

can be proved. Here\*  $x$  is an arbitrary element of the algebra, and  $r, s$  are arbitrary indices (not necessarily simple integers) denoting powers. A fuller explanation is given in § 2.

The property is invariant under linear transformations of the algebra; and if two algebras possess it, it is easily shown that their direct sum also possesses it.

Two (apparently unrelated) types of linear algebra are discussed in §§ 3, 4, and are shown to have this property. It was proved in C.T.A.† that in a commutative train algebra of rank 3 the multiplication of indices is commutative. The theorem proved in § 4 gives a wider class of algebras with this property; and using this theorem together with the results established in S.T.A., commutative train algebras of any rank, in which indices commute, can be constructed.

2. In explanation of (1.1), the conventions which were introduced in N.C. may be recalled.

In a non-associative algebra (not for the moment assumed commutative):—

(i) To obviate brackets in a product of many factors, groups of dots are placed between the factors where necessary, fewness of dots implying precedence in multiplication. For example,  $a::bc.ad^2:e$  means  $a\{[(bc)(ad^2)]e\}$ .

The *degree* of a product is the number of factors.

(ii) The product of two powers of the same element is indicated by a (generally non-commutative, non-associative) sum in the index, and a power of a power by a (generally non-commutative but associative) product

\* Concerning notation: italic letters, both small and capital, denote elements of linear algebras (hypercomplex numbers), except that the letters  $r, s, t$  denote indices of powers, and  $m, n$  denote positive integers. Greek letters denote *numbers*, i.e. elements of the field over which a linear algebra is defined; they may be thought of as ordinary real or complex numbers. Greek letters are also used as enumerating subscripts and superscripts, running from 1 to  $m$ , 1 to  $n$ , or 1 to  $\infty$ .

† I refer by initials to four papers by myself on non-associative algebra: N.C. = "On non-associative combinations", *Proc. Royal Soc. Edinburgh*, 59 (1939), 153–162; G.A. = "Genetic algebras", *ibid.*, 242–258; C.T.A. = "Commutative train algebras of ranks 2 and 3", *Journal London Math. Soc.*, 15 (1940), 136–149; S.T.A. = "Special train algebras", *Quart. J. of Math.* (in press).

in the index. For example,

$$x^{(1+(2+1))(1+2)} = (x \cdot x^2 x)^{1+2} = x \cdot x^2 x : x \cdot x^2 x : x \cdot x^2 x ;$$

$$x^{(1+2)(1+(2+1))} = (xx^2)^{1+(2+1)} = xx^2 : xx^2 : xx^2.$$

The index, evaluated as an integer in ordinary arithmetic, gives the degree of a power. But the indices as they stand may be regarded as integers of a "non-associative arithmetic," obeying the rules

$$rs \cdot t = r \cdot st, \quad r(s+t) = rs + rt; \quad (2.1)$$

but usually

$$(r+s)+t \neq r+(s+t), \quad r+s \neq s+r, \quad rs \neq sr, \quad (s+t)r \neq sr+tr.$$

If multiplication in the algebra is commutative, the indices obey also

$$r+s = s+r. \quad (2.2)$$

Powers in which the factors are absorbed one at a time are called *primary*; or *principal* if the factors are absorbed all on the right or all on the left. When multiplication is commutative, principal and primary powers coincide and are unique for a given degree. A third convention is then useful:—

(iii) The principal power of degree  $\delta$  is denoted  $x^\delta$ ; thus

$$x^4 = xx : x, \quad x^{2 \cdot 3} = x^2 x^2 \cdot x^2, \quad x^{3 \cdot 2} = x^2 x \cdot x^2 x.$$

As will be clear from the examples, equation (1.1) is not always obeyed. If it holds generally in a particular algebra, so that we can write

$$rs = sr, \quad (2.3)$$

then also, in consequence of (2.1),

$$(s+t)r = sr + tr. \quad (2.4)$$

In a commutative algebra with this property, equations (2.1, 2, 3, 4) all hold; indices therefore combine in accordance with the laws of ordinary arithmetic excepting only that addition of indices is non-associative.

The arithmetic of the indices, that is to say the totality of possible indices together with their rules of combination, may be called the *logarithmic* of an algebra. The assumptions of the commutative law of multiplication, and the special property (1.1), result in simplifications of the logarithmic; and there are algebras in which it is still further simplified. In any associative algebra, in a non-associative linear algebra



where every element satisfies a quadratic equation\*, and in Jordan's "r-number" algebras†, multiplication is associative for powers, so that

$$(r+s)+t = r+(s+t), \quad (2.5)$$

and the logarithmic becomes ordinary arithmetic. For a cyclic multiplicative group, the logarithmic is a cyclic additive group. Finally, for Boolean algebra, in which  $x^2 = x$  always, the logarithmic collapses to a single element 1.

3. Consider the linear algebra with basis  $a, b, c$  and multiplication table

$$a^2 = b, \quad b^2 = c, \quad c^2 = a, \quad bc = ca = ab = 0. \quad (3.1)$$

The general element being

$$x = \alpha a + \beta b + \gamma c, \quad (3.2)$$

we have

$$\begin{aligned} x^2 &= \gamma^2 a + \alpha^2 b + \beta^2 c, \\ x^3 &= \beta^2 \gamma a + \gamma^2 \alpha b + \alpha^2 \beta c, \\ x^4 &= x x^3 = \Sigma \alpha^2 \beta \gamma a, \quad x^{2.2} = (x^2)^2 = \Sigma \beta^4 a, \\ x^{3.2} &= (x^3)^2 = \Sigma \alpha^4 \beta^2 a, \quad x^{2.3} = (x^2)^3 = \Sigma \alpha^4 \beta^2 a, \end{aligned}$$

where  $\Sigma$  indicates summation covering cyclic permutation of  $a, \beta, \gamma$  and  $a, b, c$ .

The result  $x^4 \neq x^{2.2}$  shows that multiplication is non-associative for powers, while  $x^{3.2} = x^{2.3}$  suggests that (1.1) may hold, and this will be proved.

It will be evident that any power of  $x$  has the form  $\Sigma \alpha^{r_1} \beta^{r_2} \gamma^{r_3} a$ , since  $x$  itself has this form, and since if two such expressions are multiplied the product has the same form. Suppose then that

$$x^r = \alpha^{r_1} \beta^{r_2} \gamma^{r_3} a + \alpha^{r_3} \beta^{r_1} \gamma^{r_2} b + \alpha^{r_2} \beta^{r_3} \gamma^{r_1} c, \quad (3.3)$$

$$x^s = \alpha^{s_1} \beta^{s_2} \gamma^{s_3} a + \alpha^{s_3} \beta^{s_1} \gamma^{s_2} b + \alpha^{s_2} \beta^{s_3} \gamma^{s_1} c. \quad (3.4)$$

Then

$$x^{rs} = (x^r)^s = \Sigma (\alpha^{r_1} \beta^{r_2} \gamma^{r_3})^{s_1} (\alpha^{r_3} \beta^{r_1} \gamma^{r_2})^{s_2} (\alpha^{r_2} \beta^{r_3} \gamma^{r_1})^{s_3} a, \quad (3.5)$$

$$x^{sr} = (x^s)^r = \Sigma (\alpha^{s_1} \beta^{s_2} \gamma^{s_3})^{r_1} (\alpha^{s_3} \beta^{s_1} \gamma^{s_2})^{r_2} (\alpha^{s_2} \beta^{s_3} \gamma^{s_1})^{r_3} a. \quad (3.6)$$

\* Dickson, *Trans. American Math. Soc.*, 13 (1912), 59-73, § 5.

† Jordan, v. Neumann and Wigner, *Ann. Math.*, 35 (1934), 29-64.

It will be seen that these are equal (multiplication of the Greek letters is of course associative), and the property is proved.

Taking the above algebra to be over the field of real numbers, and applying the transformation

$$A = \lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} \nu^{\frac{1}{2}} a, \quad B = \lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} \nu^{\frac{1}{2}} b, \quad C = \lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} \nu^{\frac{1}{2}} c, \quad (3.7)$$

where  $\lambda, \mu, \nu$  are non-zero, the multiplication table (3.1) becomes

$$A^2 = \lambda B, \quad B^2 = \mu C, \quad C^2 = \nu A, \quad BC = CA = AB = 0 \quad (\lambda, \mu, \nu \neq 0). \quad (3.8)$$

It follows that the linear algebra determined by equations (3.8) has the property we are considering; and hence that the identity  $x^{rs} = x^{sr}$ , where  $x = \alpha A + \beta B + \gamma C$ , could be verified for any particular powers  $r, s$  by direct calculation of  $x^{rs}, x^{sr}$  from (3.8). This verification will continue to hold good when any of  $\lambda, \mu, \nu$  may be zero, and when the coefficient field is not necessarily the real field. Hence the property of commuting indices will still hold in these cases.

These arguments can be extended to similar algebras of order  $n$  instead of 3, so that finally we may state, without restriction of the coefficient field: *the linear algebra with multiplication table*

$$\left. \begin{aligned} a_1^2 &= \lambda_1 a_2, & a_2^2 &= \lambda_2 a_3, & \dots, & a_{n-1}^2 &= \lambda_{n-1} a_n, & a_n^2 &= \lambda_n a_1 \\ a_\theta a_\phi &= 0 & (\theta \neq \phi) \end{aligned} \right\} \quad (3.9)$$

*has the property of commuting indices.*

4. A *baric algebra* was defined in G.A. as a linear algebra possessing a non-trivial scalar representation on its coefficient field. This implies that to any element  $x$  there corresponds a number  $\xi(x)$ , called the *weight* of  $x$ , not identically zero, such that

$$\xi(x+y) = \xi(x) + \xi(y), \quad \xi(ax) = \alpha \xi(x), \quad \xi(xy) = \xi(x) \xi(y). \quad (4.1)$$

Let  $\mathbf{X}$  be a commutative baric algebra, of rank greater than 2. (If it is of rank 2, multiplication is associative for powers.) Then it was shown in C.T.A. that  $\mathbf{X}$  possesses two invariant subalgebras: the *nil subalgebra*  $\mathbf{U}$ , consisting of all nil elements (elements of zero weight, also called *u-elements*), and the *nilproduct subalgebra*  $\mathbf{P}$ , contained in or equal to  $\mathbf{U}$ , consisting of *p-elements*, or linear combinations of nilproducts; the nilproduct of two elements  $x, y$  of weights  $\xi, \eta$  being defined as

$$x \cdot y = xy - \frac{1}{2} \xi y - \frac{1}{2} \eta x. \quad (4.2)$$

Also (C.T.A., Theorems II, V, equations 7.24), by a suitable linear transformation, the basis of  $\mathbf{X}$  can be taken as

$$A, \quad u^\sigma \quad (\sigma = 1, \dots, m), \quad p^a \quad (a = 1, \dots, n), \quad (4.3)$$

where  $A$  is an arbitrarily chosen normalised element (element of weight 1), the  $u^\sigma$  are  $u$ -elements but not  $p$ -elements, and the  $p^a$  are  $p$ -elements. The multiplication table then takes the form

$$\left. \begin{aligned} A^2 &= A + (p), & Au^\sigma &= \frac{1}{2}u^\sigma + (p), & Ap^a &= (p) \\ u^\sigma u^\tau &= (p), & u^\sigma p^a &= (p), & p^a p^b &= (p), \end{aligned} \right\} \quad (4.4)$$

where the  $p$ 's in brackets denote unspecified linear combinations of the  $p^a$ . Conversely, any multiplication table of this form determines a baric algebra, whose  $p$ -elements are linear combinations of the  $p^a$ , and in which the weight of any element

$$x = \xi A + \sum \alpha_\sigma u^\sigma + \sum \beta_a p^a \quad (4.5)$$

is equal to the coefficient of  $A$ .

$$\text{If} \quad p^a p^b = 0 \quad (a, b = 1, \dots, n), \quad (4.6)$$

other product formulae being as in (4.4), then  $\mathbf{X}$  is a commutative baric algebra of a particular type, in which

$$\mathbf{P}^2 = 0,$$

i.e., the product of any two  $p$ -elements is zero. Such algebras may thus be easily constructed.

It will now be proved that in a commutative baric algebra whose nilproduct subalgebra  $\mathbf{P}$  satisfies  $\mathbf{P}^2 = 0$ , the multiplication of indices is commutative.

In the first place, let  $u$  be an element of zero weight. Disregarding the trivial case when  $r$  or  $s$  is 1,  $u^r$  is a  $p$ -element, say  $p$ , and  $p^s$  is zero, so that

$$u^{rs} = 0 = u^{sr}.$$

Next, let  $x = \xi X$  be an element of non-zero weight  $\xi$ ,  $X$  being of unit weight. Then

$$x^{rs} = \xi^{rs} X^{rs}, \quad x^{sr} = \xi^{sr} X^{sr},$$

so that  $x^{rs} = x^{sr}$  will follow from  $X^{rs} = X^{sr}$ .

It will thus be sufficient to prove the property of commuting indices for elements of unit weight. And since any element of unit weight in  $\mathbf{X}$  can be taken as the base element  $A$ , it will suffice if we can show that

$$A^{rs} = A^{sr}. \quad (4.7)$$

Let the elements  $p_1, p_2, p_3, \dots$ , be defined by the equations

$$A^2 = A + p_1, \quad Ap_\mu = p_{\mu+1} \quad (\mu = 1, 2, 3, \dots). \quad (4.8)$$

The sequence continues to infinity, though the elements so defined may be zero from some point onwards. They are all  $p$ -elements, and hence

$$p_\mu p_\nu = 0 \quad (\mu, \nu = 1, 2, 3, \dots). \quad (4.9)$$

By means of the equations (4.8, 9), any power  $A^r$  can be expressed in the form

$$A^r = A + \Sigma \alpha_\mu p_\mu. \quad (4.10)$$

(Here and below,  $\Sigma$  and  $\Sigma\Sigma$  mean respectively  $\sum_{\mu=1}^{\infty}$  and  $\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty}$ .) For  $A$  itself is of this form, and if we consider any two such expressions, say (4.10) and

$$A^s = A + \Sigma \beta_\mu p_\mu, \quad (4.11)$$

their product can be expressed in the same form, namely

$$A^{r+s} = A + p_1 + \Sigma (\alpha_\mu + \beta_\mu) p_{\mu+1}. \quad (4.12)$$

Thus, for example,

$$A^2 = A + p_1, \quad A^3 = A \cdot A^2 = A + p_1 + p_2,$$

$$A^4 = A \cdot A^3 = A + p_1 + p_2 + p_3, \quad A^{2,2} = (A + p_1)^2 = A + p_1 + 2p_2,$$

$$A^{3,2} = (A + p_1 + p_2)^2 = A + p_1 + 2p_2 + 2p_3,$$

$$A^{2,3} = (A + p_1)(A + p_1 + 2p_2) = A + p_1 + 2p_2 + 2p_3.$$

Similarly any power can be so expressed.

[Of course, the infinite sequence of  $p$ 's cannot be linearly independent, so that if the multiplication table of the algebra is known the expression of  $A^r$  linearly in  $A$  and the  $p_\mu$  will not be unique; nevertheless the expression which is derived from the equations (4.8), (4.9) for any power  $A^r$  is unique, and it is this expression which is labelled (4.10).]

Consider then any two powers  $A^r, A^s$  with the expressions (4.10), (4.11) derived from (4.8), (4.9); and let us deduce the expression for  $A^{rs}$ . This, being  $(A^r)^s$ , will be given by a formula with the same coefficients as (4.11), say

$$A^{rs} = A^r + \Sigma \beta_\mu p_\mu', \quad (4.13)$$

where the  $p_\mu'$  are defined by equations analogous to (4.8):

$$A^{r,2} = A^r + p_1', \quad A^r p_\mu' = p_{\mu+1}'. \quad (4.14)$$

These also are  $p$ -elements, so that

$$p_\mu' p_\nu' = p_\mu p_\nu' = 0. \quad (4.15)$$

$$\text{Now } A^{r.2} = (A + \sum \alpha_\mu p_\mu)^2 = A + p_1 + 2\sum \alpha_\mu p_{\mu+1}. \quad (4.16)$$

Therefore

$$\begin{aligned} p_1' &= A^{r.2} - A^r = p_1 + 2\sum \alpha_\mu p_{\mu+1} - \sum \alpha_\mu p_\mu, \\ p_2' &= A^r p_1' = (A + \sum \alpha_\mu p_\mu) p_1' = A p_1' \\ &= p_2 + 2\sum \alpha_\mu p_{\mu+2} - \sum \alpha_\mu p_{\mu+1}, \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ p_\nu' &= p_\nu + 2\sum \alpha_\mu p_{\mu+\nu} - \sum \alpha_\mu p_{\mu+\nu-1}. \end{aligned} \quad (4.17)$$

Substituting (4.10) and (4.17) in (4.13),

$$\begin{aligned} A^{rs} &= A + \sum \alpha_\mu p_\mu + \sum \beta_\mu \left\{ p_\mu + 2\sum_\nu \alpha_\nu p_{\mu+\nu} - \sum_\nu \alpha_\nu p_{\mu+\nu-1} \right\} \\ &= A + \sum (\alpha_\mu + \beta_\mu) p_\mu + 2\sum \alpha_\nu \beta_\mu p_{\mu+\nu} - \sum \alpha_\nu \beta_\mu p_{\mu+\nu-1}. \end{aligned} \quad (4.18)$$

The symmetry of this expression in  $\alpha$  and  $\beta$  shows that the same result would be obtained for  $A^{sr}$ . This proves (4.7); and, as pointed out, the property of commuting indices follows.

In conclusion, it may be pointed out that, for algebras of the type just discussed (commutative baric algebras with  $\mathbf{P}^2 = 0$ ), an extension of the property  $x^{rs} = x^{sr}$  can be proved. Let  $f(A)$ ,  $g(A)$  denote any two polynomials in  $A$  of unit weight (i.e., linear combinations of non-associative powers of  $A$  with sum of coefficients equal to unity). Each will have a characteristic expression of the form (4.10), say:

$$f(A) = A + \sum \alpha_\mu p_\mu, \quad g(A) = A + \sum \beta_\mu p_\mu; \quad (4.19)$$

and the method used above yields in place of  $X^{rs} = X^{sr}$  the identity

$$g(f(X)) = f(g(X)), \quad (4.20)$$

$X$  being any normalised element. If the polynomials are homogeneous (as for example  $\frac{1}{2}x^4 + \frac{1}{2}x^{2.2}$  is homogeneous of degree 4), we can deduce

$$g(f(x)) = f(g(x)), \quad (4.21)$$

$x$  being any element of the algebra.

Mathematical Institute,  
University of Edinburgh.



# POSTSCRIPT TO PAPERS II AND X

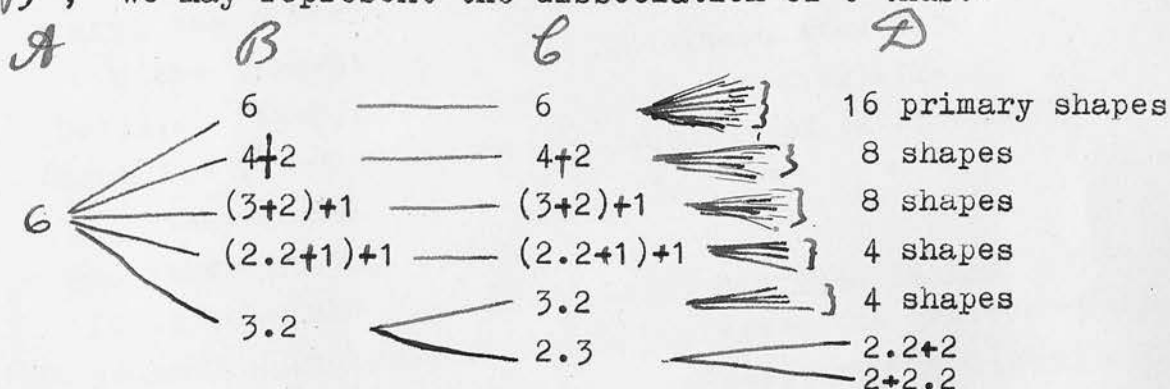
On logarithmic.

We may think of ordinary arithmetic ( $\mathcal{A}$ ) as being unfolded first into the arithmetic of commutative shapes ( $\mathcal{C}$ ) ( $\equiv$  the logarithmic of a commutative non-associative algebra), then into the arithmetic of non-commutative shapes ( $\mathcal{D}$ ) ( $\equiv$  the logarithmic of a non-commutative non-associative algebra). In this process an integer  $s$  becomes dissociated first into  $b_s$  commutative shapes, then into  $a_s$  non-commutative shapes, each commutative shape  $s$  being dissociated into  $2^{r(s)}$  non-commutative shapes. Thus the integer 6 dissociates into the six commutative shapes

6, 4+2, (3+2)+1, (2.2+1)+1, 3.2, 2.3,  
and these again into

$2^{r(6)} + 2^{r(4+2)} + \dots = 2^4 + 2^3 + 2^3 + 2^2 + 2^2 + 2 = 42 = a_6$   
elements of  $\mathcal{D}$ .

Intermediate between  $\mathcal{A}$  and  $\mathcal{C}$  is the logarithmic of a commutative non-associative algebra with commuting indices. Calling it  $\mathcal{B}$ , we may represent the dissociation of 6 thus:-



The commuting property  $rs = sr$  leads to other relations, so that elements of  $\mathcal{B}$  which at first sight appear to be prime may in fact be factorisable. For example

$$\begin{aligned}
 6.2 &= (5+1)2 = 5.2+2 \\
 &= (4+1)2+2 = (4.2+2)+2 \\
 &= \{(3+1)2+2\}+2 = \{(3.2+2)+2\}+2 \\
 &= [\{(2+1)2+2\}+2]+2 = [\{(2.2+2)+2\}+2]+2 = 2.6.
 \end{aligned}$$

So the five shapes

6.2, 5.2+2, (4.2+2)+2,  $\{(3.2+2)+2\}+2$ , 2.6,  
which are distinct in  $\mathcal{C}$ , are equal in  $\mathcal{B}$ ; in other words, the product 6.2 in  $\mathcal{B}$  becomes dissociated into 5 distinct shapes in  $\mathcal{C}$ , of which 3 are prime.

It is interesting to consider what properties of ordinary integers are "dissociable", i.e. are passed on to one or other of the corresponding non-associative integers. E.g. the property of being prime would be passed on to all corresponding non-associative integers; that of being composite only to a few. Considering the transition from  $\mathcal{A}$  to  $\mathcal{C}$ , and using an arrow to denote "is passed on to", the following are seen to be dissociable properties of the ordinary integer 6:-

6 is generated by adding 1 to 5  $\rightarrow 6$ ,  $(3+2)+1$ ,  $(2.2+1)+1$ .

6 is a multiple of 3  $\rightarrow 3.2 = 3+3$ .

6 is a multiple of 2  $\rightarrow 2.3 = (2+2)+2$ .

6 is the sum of three squares  $\left\{ \begin{array}{l} \rightarrow (2.2+1)+1 = (2^2+1^2)+1^2, \\ \rightarrow 2.3 = (1^2+1^2)+2^2. \end{array} \right.$

It more often happens that a property is itself dissociated into several properties. Thus in  $\mathcal{A}$  the properties of

(i) being of the form  $n(n+1) = n^2+n$ ,

(ii) " " " "  $-(n+1)n$ ,

(iii) " a triangular number,

are indistinguishable, and are possessed by 6; whereas in  $\mathcal{C}$

(i)  $\rightarrow 2.3 = 2^2 + 2$ , (ii)  $\rightarrow 3.2$ , (iii)  $\rightarrow 3.2 = (1+2)+3$ ,  $(3+2)+1$ .

Or again, the property of being a multiple of 3 is possessed in  $\mathcal{C}$  by 3.2. In the transition to  $\mathcal{D}$ , 3.2 dissociates into

$(2+1)2$ ,  $(1+2)2$ ,  $(2+1)+(1+2)$ ,  $(1+2)+(2+1)$ ,

of which the first receives the property of being a multiple of 2+1, the second " " " " " " " " 1+2, and the other two are prime.

It is possible that there exist non-commutative linear algebras with commuting indices, in which therefore the indices would obey:-

$$rs = sr, \quad rs.t = r.st, \quad r(s+t) = rs + rt = (s+t)r,$$

$$r+s \neq s+r, \quad (r+s)+t \neq r+(s+t).$$

Whether this is so or not, the abstract non-associative arithmetic with these properties certainly exists, and like  $\mathcal{C}$  it is intermediate between  $\mathcal{B}$  and  $\mathcal{D}$ .

The logarithmic of a non-associative algebra of  $s$  dimensions (see Postscript to Part One) would be a non-associative arithmetic of  $s$  dimensions, representing the indices of powers. One may conjecture that there exist non-associative linear algebras of  $s$  dimensions with a property analogous to that of commuting indices.